# Special Solutions on Alternative Gravitation Theories in The Field of Cosmology

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# SPECIAL SOLUTIONS ON ALTERNATIVE GRAVITATION THEORIES IN THE FIELD OF COSMOLOGY

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# Chapter 1

## DYNAMICS OF THE UNIVERSE UNDER F(R,G,T) THEORY WITH A SPECIALIZED DECELERATION PARAMETER

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## I. INTRODUCTION

Findings based on a range of observational data, including supernova type Ia, cosmic microwave background (CMB) radiation, and large-scale structures, has shown that the cosmos is undergoing a phase of increasingly rapid expansion (Perlmutter et al., 1999; Riess et al., 1998). General Relativity (GR), while successful in many areas, fails to fully explain this phase of accelerated expansion, which is attributed to an unknown entity termed dark energy (DE). DE exerts a repulsive force through its negative pressure and is responsible for this cosmic acceleration. Alternative gravitational theories have been suggested as a promising framework to explore the properties of DE, as they introduce changes on the traditional Einstein-Hilbert action.

One popular approach within these alternative theories is $f(R)$ tehory, in which the Einstein-Hilbert action is modified by extending the Ricci scalar $R$ to an arbitrary function of $R$ (De Felice and Tsujikawa, 2010; Amendola et al., 2007). This modification has been successful in explaining the acceleration of the universe (Carroll et al., 2004). Capozziello et al. (2006) introduced a more consistent theory within the framework of $f(R)$ gravity that satisfies Newtonian limits and proposed conditions for constructing viable cosmological models.

Another widely studied alternative theory is Gauss-Bonnet (GB) gravity, which modifies the Einstein-Hilbert action by incorporating the GB invariant, $\mathcal{G} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$, where $R,\ R_{ab}$ and $R_{abcd}$ are Ricci skalar, Ricci and Riemann tensors, respectively. GB gravity is appealing because it avoids second-order ghost instabilities (Mavromatos, 2022; Nojiri and Odintsov, 2005). Nojiri and Odintsov (2011) further extended this by introducing $f(\mathcal{G})$ gravity, a theory where the GB term is generalized to an arbitrary function. This approach is a compelling alternative for explaining DE, and it has been explored for its ability to describe both the universe's evolution during its initial stages and its accelerated expansion in the present era (Cognola et al.,2006).

Recent studies have focused on combining the aforementioned theories with other components like matter, leading to frameworks like $f(R,T)$ gravity, in which $T$ denotes the trace of the stress-energy tensor (Harko et al., 2011).

These theories introduce additional interactions between geometry and matter, further refining our understanding of cosmic acceleration. Another extension, $f(R,\mathcal{G},T)$ gravity, was setted by considering a general function of $R,\mathcal{G}$, and $T$, combining the dynamics of Ricci curvature, Gauss-Bonnet terms, and matter interactions (Debnath, 2020). This model presents a novel approach for studying both the evolution of the universe and the behavior of compact astrophysical objects.

The objective of this work is to investigate the implications of $f(R,\mathcal{G},T)$ theory in describing the accelerated growth of the universe, especially focusing on the dynamics of the deceleration parameter. We also aim to compare observational data with theoretical predictions and explore the cosmological consequences of this theory.

This study is structured as follows. In Section II, we give a brief review of the $f(R,G,T)$ gravity theory, focusing on its theoretical and mathematical frameworks. Section III presents the observational constraints applied to the model. In Section IV, we explore the cosmological implications of the model. Finally, Section V concludes the paper by summarizing the results and including the potential applications of the model to other cosmological observations.

## II. REVIEW OF $f(R,\mathcal{G},T)$ GRAVITY

The $f(R,\mathcal{G},T)$ theory broadens the scope of modified gravity models. This framework enables a deeper interaction between matter and geometry, offering a robust approach to account for several cosmological problems, such as the cosmic acceleration of the universe. The action for this modified theory is written as (Debnath, 2020)

$$\mathcal{S} = \int \left(\frac{1}{2}f(R,\mathcal{G},T) + \mathcal{L}_M\right)\sqrt{-g}d^4x \tag{1}$$

The geometric terms that appear in the action are defined as follows:

$$R = g^{ab}R_{ab} \tag{2}$$

$$\mathcal{G} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} \tag{3}$$

$$T = g^{ab}T_{ab} \tag{4}$$

This theory of gravity provides a generalization of Einstein's field equations in the following manner (Debnath, 2020):

$$\left(R_{ab} - \frac{1}{2}g_{ab}R\right)f_R + g_{ab}\nabla^2 f_R - \nabla_a\nabla_b f_R + \big(2RR_{ab} - 4R_{ac}R_b^c - 4R_{acbd}R^{cd} + 2R_{acd\rho}R_b^{cd\rho}\big)f_{\mathcal{G}} = T_{ab} - (T_{ab} - \Theta_{ab})f_T \tag{5}$$

where $f_R = \frac{\partial f}{\partial R}$, $f_{\mathcal{G}} = \frac{\partial f}{\partial \mathcal{G}}$ and $f_T = \frac{\partial f}{\partial T}$. Here, $\Theta_{ab}$ represents an additional term related to the matter Lagrangian, and $\nabla_a$ denotes covariant differentiation.

Taking the covariant divergence of equation (5) yields

$$\nabla^a T_{ab} = \frac{f_T}{(1-f_T)}\left[(T_{ab} + \Theta_{ab})\nabla^a \ln f_T + \nabla^a\Theta_{ab} - \frac{1}{2}g_{ab}\nabla^a T\right] \tag{6}$$

It is evident that this last expression does not depend on the functional forms of $f_R$ and $f_G$. Moreover, here $\Theta_{ab}$ is defined as

$$\Theta_{ab} = 2T_{ab} + g_{ab}\mathcal{L}_m - 2g^{\alpha\beta}\frac{\partial^2\mathcal{L}_m}{\partial g^{ab}\partial g^{\alpha\beta}} \tag{7}$$

If $\mathcal{L}_m$ is known, we can determine $\Theta_{ab}$. In this framework, for a perfect fluid $T_{ab}$ is typically expressed as:

$$T_{ab} = (\epsilon + p)v_a v_b + p g_{ab} \tag{8}$$

Here, $\epsilon$ represents the energy density, $p$ denotes the pressure, and $v_a$ is the fluid's four-velocity, which satisfies the condition $v_a v^a = -1$. The term $\Theta_{ab}$ is defined as:

$$\Theta_{ab} = -2T_{ab} + p g_{ab}. \tag{9}$$

In applying this theory to cosmology, we use the Friedmann-Robertson-Walker (FRW) model. For a flat geometry, this metric is given by:

$$ds^2 = -dt^2 + S^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{10}$$

Here, $S(t)$ represents the scale factor, and the Hubble parameter $H$, which characterizes the universe's expansion rate, is defined as:

$$H = \frac{\frac{dS}{dt}}{S} \tag{11}$$

In this context, the Ricci scalar and Gauss-Bonnet term can be obtained as in the following forms:

$$\frac{R}{6} = \left(2H^2 + \frac{dH}{dt}\right) \tag{12}$$

$$\frac{\mathcal{G}}{24} = H^2\left(H^2 + \frac{dH}{dt}\right) \tag{13}$$

and the trace $T$ of $T_{ab}$ for a perfect fluid becomes:

$$T = 3p - \epsilon \tag{14}$$

The modified field equations of this gravity are derived by

$$3H^2 = \frac{1}{f_R}\left[\epsilon + (\epsilon + p)f_T + \frac{1}{2}(Rf_R - f) - 3H\frac{df_R}{dt} + 12H^2\left(\frac{dH}{dt} + H^2\right)f_{\mathcal{G}} - 12H^3\frac{df_{\mathcal{G}}}{dt}\right] \tag{15}$$

$$2\frac{dH}{dt} + 3H^2 = -\frac{1}{f_R}\left[p - \frac{1}{2}(Rf_R - f) + 2H\frac{df_R}{dt} + \frac{d^2f_R}{dt^2} - 12H^2\left(\frac{dH}{dt} + H^2\right)f_{\mathcal{G}} + 8H\left(\frac{dH}{dt} + H^2\right)\frac{df_{\mathcal{G}}}{dt} + 4H^2\frac{d^2f_{\mathcal{G}}}{dt^2}\right] \tag{16}$$

On the other hand, equation (6) leads to the following modified conservation equation which can be regarded as a non-conservation equation

$$\frac{d\epsilon}{dt} + 3H(\epsilon + p) = \frac{1}{2}\frac{dT}{dt} - p\frac{df_T}{dt}(\epsilon + p) - f_T \tag{17}$$

In this work, we use a particular form of this theory as follows

$$f(R, \mathcal{G}, T) = \mathcal{G}^2 + bRT \tag{18}$$

where $b$ is a constant. The functional form $f(R, \mathcal{G}, T) = \mathcal{G}^2 + bRT$ offers several advantages. The $\mathcal{G}$ term allows us to capture higher-order curvature effects and contributes to stabilizing the model by avoiding ghost instabilities. Meanwhile, the $RT$ coupling introduces direct interactions between matter and geometry, enriching the model's capacity to describe both the expansion history and matter dynamics of the universe. This combination provides a versatile framework to investigate shifts from the conventional ΛCDM model and better understand cosmic acceleration.

Consequently, by adopting the form of this theory, the field equations simplify to

$$3bH^2T = \epsilon + b(\epsilon + p)R - \frac{\mathcal{G}^2}{2} - 3bH\frac{dT}{dt} + 24H^2\left(\frac{dH}{dt} + H^2\right)\mathcal{G} - 24H^3\frac{d\mathcal{G}}{dt} \tag{19}$$

$$b\left(2\frac{dH}{dt} + 3H^2\right)T = p + \frac{\mathcal{G}^2}{2} + 2nH\frac{dT}{dt} + b\frac{d^2T}{dt} - 24H^2\left(\frac{dH}{dt} + H^2\right)$$
$$+16H\left(\frac{dH}{dt} + H^2\right)\frac{d\mathcal{G}}{dt} + 8H^2\frac{d^2\mathcal{G}}{dt^2} \tag{20}$$

where $\frac{dT}{dt} = 3\frac{dp}{dt} - \frac{d\epsilon}{dt}$ and $\frac{d^2T}{dt^2} = 3\frac{d^2p}{dt^2} - \frac{d^2\epsilon}{dt^2}$. As observed, the last equation incorporates not only $\epsilon$ and $p$ but also their first and second time derivatives. To address this mathematical challenge, we assume a barotropic equation of state (EoS) $p = \omega\epsilon$, which enables us to reformulate equation (17) as

$$\frac{d\epsilon}{dt} = \frac{2(\omega+1)\left(b\frac{dR}{dt}+3H\right)}{(\omega-1)bR-2}\epsilon \tag{21}$$

Here, $\omega$ is the EoS parameter. Differentiating the last equation with respect to time gives

$$\frac{d^2\epsilon}{dt^2} = \left[\frac{4\left[(\omega+1)\left(b\frac{dR}{dt}+3H\right)\right]^2}{[(\omega-1)bR-2]^2} + \frac{2(\omega+1)\left(b\frac{d^2R}{dt^2}+3\dot{H}\right)}{[(\omega-1)bR-2]} - \frac{2\left[(\omega+1)\left(b\frac{dR}{dt}+3H\right)\right](\omega-1)b\frac{dR}{dt}}{[(\omega-1)bR-2]^2}\right]\epsilon \tag{22}$$

We can extract the energy density from equations (21) and (22) to obtain

$$\epsilon = \frac{b(3\omega-1)\left(\frac{d\epsilon}{dt}-\frac{d^2\epsilon}{dt^2}\right)+8H\frac{d\mathcal{G}}{dt}\left(H^2-2\frac{dH}{dt}\right)-8H^2\frac{d^2\mathcal{G}}{dt^2}}{3b(\omega-1)\dot{H}^2+(bR+1)(\omega+1)} \tag{23}$$

To find a solution for the field equations, in this work, we employ a special form of the deceleration parameter $q$. The definition of $q$ is given as

$$q = -\frac{\frac{d^2 S}{dt^2}}{H^2 S} = -\left(\frac{\dot{H}}{H^2} + 1\right) \tag{24}$$

We now choose $q$ as (Al Mamon and Das, 2016):

$$q = n + m\frac{z(1+z)}{1+z^2} \tag{25}$$

since the parametrization allows for a smooth shift between decelerating and accelerating eras, accurately reflecting the observational out put of the historical expansion of the universe. Additionally, it is flexible enough to model different cosmic epochs, making it suitable for fitting observational data over a wide range of redshifts. By using this form, we gain better control over how the deceleration parameter evolves with redshift $z$, facilitating comparisons with theoretical models and standard cosmology.

Next, we attempt to determine most suitable values for the parameters $n$ and $m$ by comparison with observations. From equations (24) and (25), using $H = -\frac{1}{1+z}\frac{dz}{dt}$, we find $H(z)$ as:

$$H = H_0(1+z)^{(1+n)}(1+z^2)^{\frac{m}{2}} \tag{26}$$

## III. OBSERVATIONAL CONSTRAINTS

To assess the viability of any cosmological model, comparing theoretical predictions with observational results is essential. For this purpose, several datasets are typically used, including data from Cosmic Chronometers (CC), and Pantheon sample of Type Ia supernovae. By imposing limits on the model parameters using these datasets, we can evaluate how well this gravity model fits with the observed universe. We can assess how well this gravity model aligns with the observed universe or develop a cosmological model that better fits the observational data.

### A. Cosmic Chronometer (CC) datasets

The Cosmic Chronometer (CC) datasets are essential for determining the Hubble rate through the use of ancient galaxies with minimal evolutionary changes, observed across narrow redshift intervals (Carroll et al., 2004; Wang and Mota, 2020; Nunes et al., 2016). This approach relies on differential aging to assess the age differences between these galaxies at varying redshifts. By applying the expression $H = -\frac{1}{1+z}\frac{dz}{dt}$ for the Hubble parameter, the CC datasets allow for an unbiased determination of $H_0$ (the current Hubble constant). To conduct the Monte Carlo Markov Chain (MCMC) analysis, a customized chi-square function specifically tailored for cosmic chronometer data is used:

$$\chi^2 = \sum_{k=1}^{31} \frac{H_{th}(z_k) - H_{obs}(z_k)}{d_H^2(z_k)} \tag{27}$$

where $H_{th}(z_k)$ and $H_{obs}(z_k)$ are the Hubble parameter and observed Hubble parameter, respectively, and $d_H^2(z_k)$ represents the measurement uncertainties associated with the observed values of $H_{obs}(z_k)$.

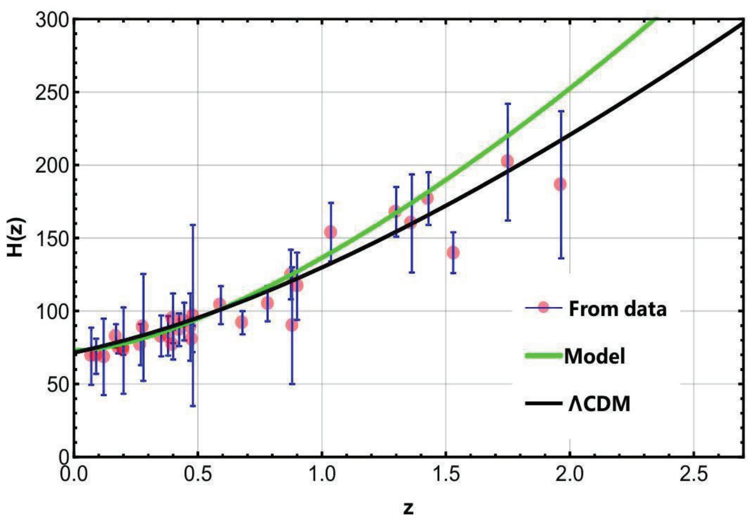


FIG. 1. $H(z)$ vs $z$.

The Figure 1 presents the the change of the Hubble parameter $H(z)$ with redshift $z$. The pink points with error bars represent observational data, illustrating the measured values of $H$ at different $z$ intervals, with uncertainties. The green curve corresponds to the forecasts of the proposed cosmological model, while the black curve displays the standard $\Lambda$ CDM model. The strong correspondence of the black curve with the observational data indicates consistency with the standard cosmological model for low to intermediate redshifts. However, the deviations at higher redshifts (as shown by the differences between the green and black curves) may suggest possible tensions or novel physics extending beyond the standard ΛCDM paradigm. This comparison is crucial for evaluating the reliability of cosmological models and understanding the expansion history of our universe.

## B. Pantheon datasets

This study utilizes a dataset comprising 1048 Type Ia supernovae (SNe Ia) luminosity distance predicts drawn from several sources, all integrated within the Pantheon sample (Scolnic et al., 2018; Chang et al., 2019). In our analysis of cosmological observational data, we employ the MCMC sampling method, extending prior research by incorporating a more comprehensive dataset and imposing stricter priors on model parameters. Especially, the focus is placed on key parameters, with the MCMC sampling performed using the emcee library (Foreman-Mackey et al., 2013), ensuring parallel processing with 100 walkers over 1,000 steps for robust and reliable outcomes.

$$\chi^2_{\text{Pan}} = \sum_{i,j=1}^{1048} \Delta\mu_i \left(C^{-1}_{\text{Pan}}\right)_{ij} \Delta\mu_j \tag{28}$$

In this context, $\Delta\mu_i$ denotes the difference between the theoretical distance modulus $\mu_{th}$ and the observed distance modulus $\mu_{obs}$. Additionally, $C^{-1}_{Pan}$ denotes the inverse of the covariance matrix associated with the Pantheon data. To get in-depth information, references (Foreman-Mackey et al., 2013) provide valuable insights.

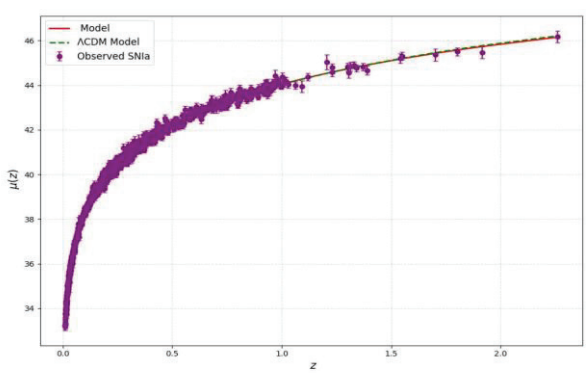


FIG. 2. $\mu(z)$ vs $z$.

The Figure 2 gives the comparison between the distance modulus $\mu(z)$ and the redshift $z$ for observed SNe Ia data and two theoretical models. The distance modulus, $\mu(z)$, is related to the luminosity distance. The purple data points with error bars represent the observed SNe Ia, showing the measured distance moduli with associated uncertainties. The red solid curve denotes the predictions from the proposed model, while the green dashed line denotes the standard $\Lambda$ CDM model.

The Figures 3 and 4 illustrate the joint posterior distributions and marginalized uncertainties for three parameters $m, H_0$, and $n$ of our model, obtained from CC and SNIa data, respectively. The parameter $m$ and $n$ are the model parameters and $H_0$ is the Hubble constant. Each diagonal plot shows the marginalized probability distribution for a single parameter with the mean and standard deviation (e.g., $m = 0.53 \pm 0.16, H_0 = 66.3^{+2.7}_{-3.1}$, and $n = -0.39 \pm 0.15$ as shown in Figure 3 and $m = 1.01 \pm 0.18, H_0 = 72.21 \pm 0.32$, and $n = -0.649 \pm 0.058$ as shown in Figure 4). The off-diagonal panels represent 2D confidence regions (e.g., 68% and 95% credible intervals) indicating correlations between parameter pairs. The green shaded regions highlight these constraints, showing that the parameters are not completely independent but exhibit correlations-most notably between $m$ and $n$. The close alignment along sloped contours suggests a degeneracy between the two, meaning that varying one parameter could partially compensate for variations in the other to fit observational data. The close agreement between the models and the observed data in Figures 3 and 4, especially at low and intermediate redshifts, suggests that the theoretical model provides a reasonable fit to the observations. However, subtle deviations at higher redshifts could indicate the need for more refined models or suggest new physics beyond the standard cosmological framework.

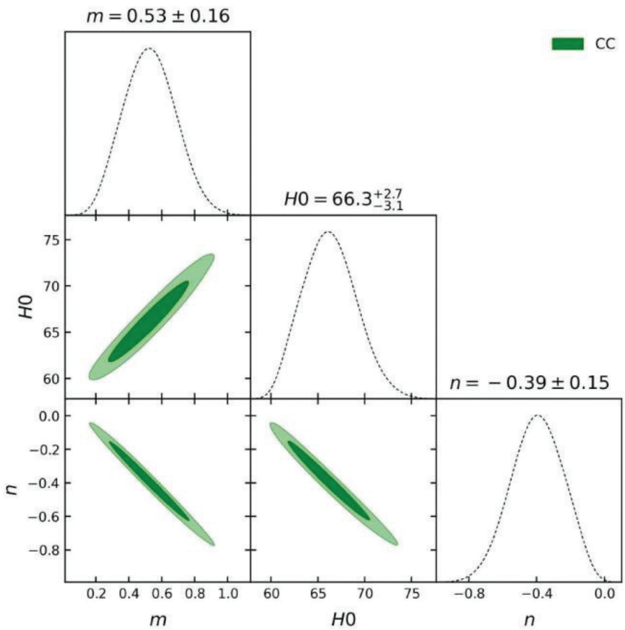


FIG. 3. Constraints on the parameters $m, H_0$, and $n$ obtained from CC

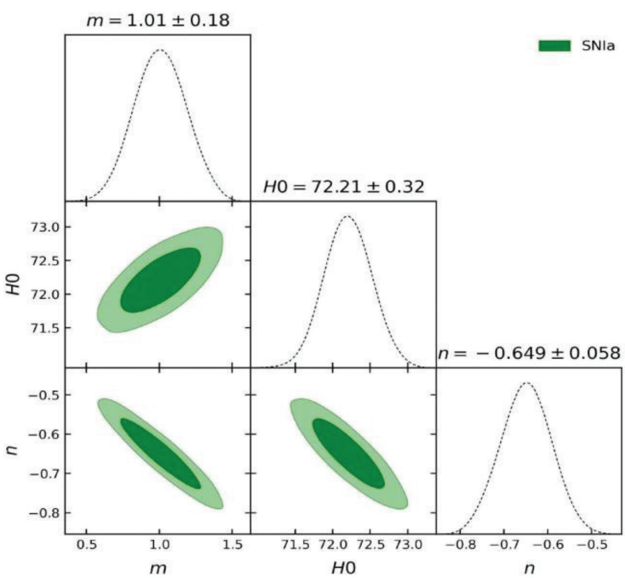


FIG. 4. Constraints on the parameters $m, H_0$, and $n$ obtained from SNIa

## IV. COSMOLOGICAL IMPLICATIONS OF THE MODEL

The deceleration parameter $(q)$ is an essential cosmological quantity that offers an understanding of the pace at which the expansion of the universe is slowdown or speedup. A positive $q$ corresponds to a slowdown universe, whereas a negative $q$ reflects speedup universe. Indicate that the universe is presently undergoing an accelerating phase, implying a negative value for the deceleration parameter. However, at earlier epochs, it is believed that the universe was in a decelerating phase, making $q$ positive at that time. Understanding the progression of the deceleration parameter over time helps us trace the history of cosmic expansion and its transition between decelerating and accelerating phases.

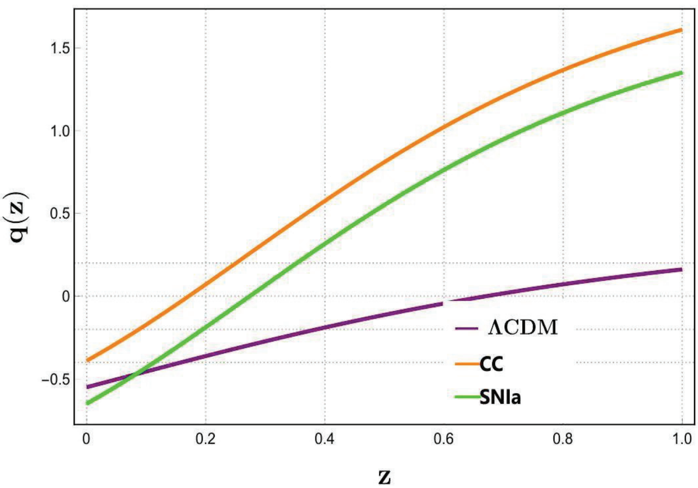


FIG. 5. $q(z)$ vs $z$

The Figure 5 displays the variation of the deceleration parameter $q(z)$ with redshift $z$ for our two models one is constrained by CC (orange line) and the other one is constrained by SNIa (green line) in comparison with ΛCDM model (purple line). Comparing these models helps test the consistency of our theoretical models with the standard cosmological model. The plot shows that our models predict a change from a slowing expansion to a speeding one, matching the well-known rapid expansion of the universe.

Although the trajectories of our models behave very closely to those of the ΛCDM model at lower redshift range, differences emerge at high redshifts.

This may imply a different behavior of dark energy or alternative cosmological dynamics in our models.

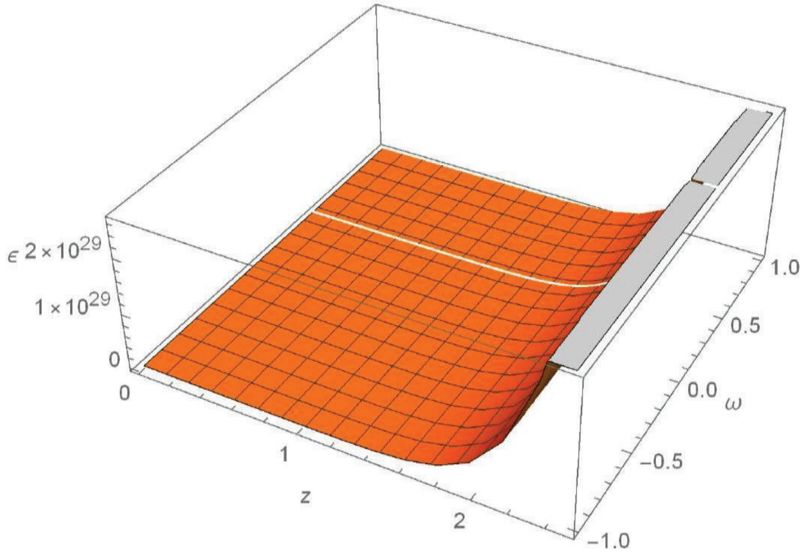


FIG. 6. $\epsilon(z)$ vs $z$ for CC data

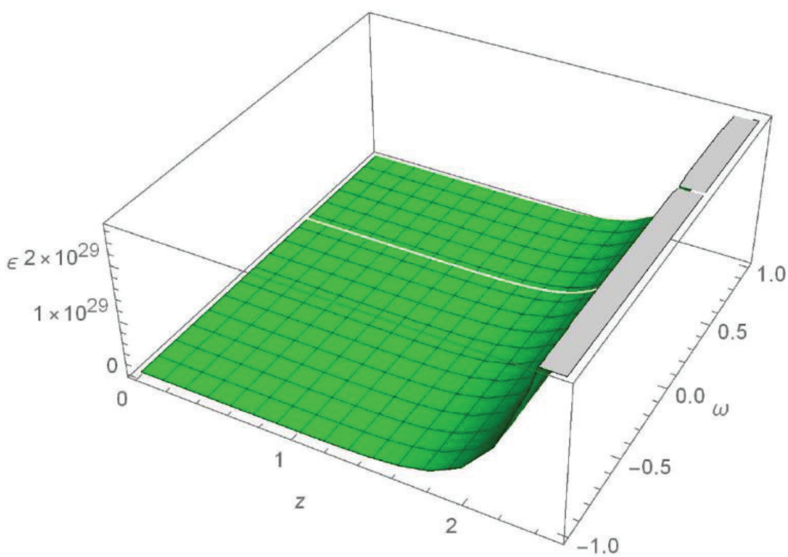


FIG. 7. $\epsilon(z)$ vs $z$ for SNIa data

The Figure 6 and 7 show a 3D plot of the variation of the energy density $\epsilon(z)$ with $z$ and the equation-of-state parameter $\omega$ for our two constrained models abtained from CC and SNIa datasets, respectively. The parameter $\omega$ governs the ratio between the pressure and energy density of the cosmological fluid, as defined above, with typical values such as $\omega = 0{,}1/3, -1$ for matter, for radiation, for a cosmological constant ($\Lambda$), respectively. The surfaces in the 3D plot indicate how the energy density evolves over time for different values of $\omega$. As redshift increases, the energy density generally rises in both two models, due to the shrinking volume of the universe. The slope of the surface depends on $\omega$, with steeper slopes corresponding to stiffer equations of state (such as radiation) and flatter slopes indicating dark energy-like behavior.

## V. CONCLUSION

In this study, we explored the dynamics of the universe in context of the $f(R, \mathcal{G}, T)$ theory, focusing on a specialized form of the deceleration parameter $q(z)$. This extended model, which integrates the Ricci scalar $R$, the Gauss-Bonnet invariant $G$, and the trace of the energy-momentum tensor $T$, offers a comprehensive approach to understanding both cosmic acceleration and matter-geometry interactions. Our work demonstrates that the selected functional form $f(R, \mathcal{G}, T) = \mathcal{G}^2 + bRT$ provides a robust framework for capturing higher-order curvature effects while allowing direct coupling between matter and geometry, enriching our understanding of cosmic evolution beyond the standard ΛCDM paradigm.

To test the theoretical predictions, we applied observational constraints using two independent datasets: Cosmic Chronometers (CC) and the Pantheon sample of Type Ia supernovae (SNe Ia). By employing Monte Carlo Markov Chain (MCMC) analysis, we determined the optimal values for the parameters $m, n$, and $H_0$, which govern the evolution of the deceleration parameter and the Hubble parameter. The results reveal that the proposed model closely aligns with observational data at low and intermediate redshifts, capturing the shift from slowed expansion to accelerated growth. However, subtle divergences from the standard ΛCDM model at higher $z$ were observed, suggesting the potential need for refined models or new physics to better describe cosmic acceleration at those epochs.

The variation of the deceleration parameter $q(z)$ as a function of redshift was also analyzed. Our findings confirm that both the CC and SNe Ia datasets predict a transition from a decelerating to an accelerating universe, consistent with the observed accelerated expansion. The comparison between these models and the standard $\Lambda$ CDM model highlights minor differences, particularly at high redshifts, where alternative cosmological dynamics may come into play. This suggests that the $f(R, \mathcal{G}, T)$ gravity model offers additional flexibility in modeling the universe's expansion history compared to the standard framework.

Moreover, the energy density evolution for values of $\omega$ in the range of $(-1,1)$ was investigated using 3D plots. These visualizations demonstrated how the energy density evolves over time, with steeper slopes corresponding to radiation dominated eras and flatter slopes indicating dark energy-like behavior. This analysis provided further insights into the role of different cosmic components across various epochs. It would also be appropriate to point out that the energy density always remains positive within the given $\omega$ range.

As a final remark, we can conclude that the $f(R, \mathcal{G}, T)$ gravity model, with its specialized deceleration parameter, offers an alternative framework for studying the universe's expansion. The ability to capture both matter-geometry interactions and higher-order curvature effects makes this model particularly versatile. Future work will involve extending the analysis to include additional observational data such as BAO and Phanteon+ measurements, to further constrain the model parameters and test its predictions across different cosmic epochs.

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# Chapter 2

## STRING FLUID DISTRIBUTION IN F(R,Φ,X) GRAVITY


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## 1. INTRODUCTION

Cosmological observations in recent years have revealed some results that the General Relativity Theory could not explain or complete regarding the early periods of the universe. As a result, in addition to evolution of ideas created for the early periods, there has also been a rapid development in gravitational theories. Despite all these developments, a theory that can fully explain that the early periods of the universe have not yet emerged. However, some hybrid and modified theories promise. It is assumed that the universe collapsed into a single point. Time and space were formed at an incredible speed with the Big Bang and the universe began to evolve at the same speed. Some topological defects occurred during this process of the universe. These topological defects caused some symmetry breaks. Symmetry breaks have revealed some structures that scientists cannot yet fully explain. The first of these structures is monopole. Immediately afterwards, the one-dimensional string emerged. Later, two-dimensional structures called domain walls emerged. Latest structures are three and more dimensional textures. Explaining the formation mechanisms of these structures in full is necessary to determine some uncertainty subjects on the universe. In addition, it is obvious that these structures can help explain the matter, known and/or unknown dark matter. Therefore, studies on alternative/modified gravitational theories also focus on these issues (Özemre, 1982; Asmodelle, 2017).

This study examines the string fluid matter which fills the universe. Also, we interest $f(R,\Phi,X)$ gravity which is a hybrid modified gravitational theory. The theory has advantages since the results obtained are formed from a hybrid theory, they can be reduced to many sub-gravity theories in limiting cases. For all these reasons, studies on modified gravitational theories are among the favorite topics of cosmology. General Relativity presents that the behavior of space and time depends on a scalar called the Ricci Scalar (Einstein, 1915). However, observations with contribution of technology revealed that the Ricci Scalar does not fully meet this behavior. With this motivation, scientists quickly began to produce new theories. One of the most resounding of these is the $f(R)$ gravitational theory (Buchdahl, 1970). The theory defined a function depending on a function instead of the Ricci Scalar. By adding the scalar potential to this function, $f(R,\Phi)$ gravitation theory emerged (Hwang, 1990). In addition, with the development of quantum cosmology, k-essence theory emerged, which was derived with a kinetic term depending on the scalar potential and this potential (Chiba *et al.*, 2000). Finally, $f(R, \Phi, X)$ gravitation theory emerged (Hwang and Noh, 2002; Tsujikawa, 2007). The strongest aspect of this theory is the variety and comprehensiveness of the models that can be selected.

Considering unstable D-branes and classical dynamics, Gibson *et al.* (2001) discussed the dynamics of a massive relativistic string fluid in a real vacuum, with Hamiltonian dynamics and vanishing action on freely moving electric flux lines at the extreme point of the tachyon condensate. They investigated possibility that some of dark matter is not composed of ordinary cold dark matter (CDM) dust-like particles, but is in form of string fluid whose barotropic factor of cosmic origin. To this end, Capozziello *et al.* (2006) investigated dynamics of a spatially flat universe by splitting the dark matter density parameter into two terms and modeling the CDM, a fluid of baryons, strings, and dark energy, as cosmological constant or negative-pressure fluid with a steady-state equation $w < 0$. Kim *et al.* (2003) presented rolling tachyon with constant electric and magnetic fields for the Born-Infeld type unstable D3-brane system. Pawar and Dabre (2023) discussed Bianchi-VI universe with perfect fluid including string cloud within scope of $f(R, T)$ gravitation theory. Some physical and kinematic parameters of the model were created by exact solutions of the field equations.

This study is planned as follows: In Section 2, by defining the string fluid in Friedmann-Robertson-Walker space-time, solutions of field equation solutions were obtained in $f(R, \Phi, X)$ theory. In addition, some useful graphs were examined for the obtained parameters. In Section 3, all results obtained with the selected model were interpreted.

## 2. STRING FLUID SOLUTIONS in $f(R, \Phi, X)$ GRAVITY

$f(R, \Phi, X)$ theory emerged as a result of the work of Hwang and Noh (2002). Firstly, this theory is a scalar-tensor theory. This theory contains, $R$ Ricci Scalar, $\Phi$ scalar potential and $X$ kinetic term depending on this scalar potential, respectively (Bahamonde et al., 2015). This theory has the ability to be reduced to hybrid models such as $f(R)$ theory and k-essence theory in order to study ideas that have become popular in recent years such as dark energy. Therefore, it is a theory that has potential to take important steps in examining late-time dilation and $\Lambda - CDM$ evolution.

Action function of $f(R,\Phi,X)$ theory is generalized as(Tsujikawa, 2007):

$$S=\frac{1}{16\pi G}\int\sqrt{-g}\,f(R,\Phi,X)d^4x+S_m \qquad (1)$$

here Einstein-Hilbert action integral defines the space-time curvature, the scalar potential and the kinetic term related to it. Kinetic term $(X)$ is described as follows:

$$X(\Phi)\propto\frac{1}{2}\left[\Phi^{,\alpha}\Phi_{,\alpha}\right]\varepsilon \qquad (2)$$

where, $\varepsilon$ is parameter. It can be commonly chosen as $\varepsilon=1$ in solutions of field equation (Tsujikawa, 2007). One can obtain field equation for $f(R,\Phi,X)$ theory from the variation of Equation (1) in the following form (Bahamonde et al., 2015):

$$FG_{ik}-\frac{1}{2}(f(R,\Phi,X)-RF)g_{ik}-\nabla_i\nabla_kF+g_{ik}\nabla_\alpha\nabla^\alpha F-\frac{\xi}{2}H(\nabla_i\Phi)(\nabla_k\Phi)=T_{ik} \qquad (3)$$

here $F\equiv\frac{df}{dR}$ and $H\equiv\frac{df}{dX}$. In $f(R,\Phi,X)$ theory, another substantial equation, Klein-Gordon equation, is served by:

$$\nabla_i\left(H\nabla^i\Phi\right)+\varepsilon N=0, \qquad (4)$$

where $N\equiv\frac{df}{d\Phi}$ (Tsujikawa, 2007).

The choice of Friedmann-Robertson-Walker (FRW) space-time is important to examine the last period of the universe in particular. In addition, when some parameters are taken into account, it can be easily reduced to the first period of the universe, and FRW universe makes an important contribution to understanding all periods of the universe. FRW space-time is defined as follows, considering Cartesian coordinates:

$$ds^2 = A^2(t)dx^2 + A^2(t)dy^2 + A^2(t)dz^2 - dt^2. \tag{5}$$

The energy-momentum tensor for string fluid is identified in the following form:

$$T_{ik} = (q + \rho)(u_i u_k - x_i x_k) + q g_{ik}. \tag{6}$$

String fluid can be formed as equation (6). In this case, $\rho$ refers to energy density of string fluid and $q$ refers to tendency of string. Also, $u_i$ and $x_i$ are the four-velocity vectors. In this study, observer with co-moving motion was taken into consideration.

Ricci Scalar is found by considering Equation (5):

$$R = \frac{6A'^2 + 6AA''}{A^2}. \tag{7}$$

Field Equations and Klein-Gordon equation for FRW type string fluid are found in $f(R, \Phi, X)$ theory as follows:

$$-A^2\left(\varepsilon H\Phi'^2 + 2F'' - 4(\rho + q)\right) + 2A(F'A' - 2FA'') + 4FA'^2 = 0, \tag{8}$$

$$-A^2\left(\varepsilon H\Phi'^2 - 2F'' + 4(\rho + q)\right) + 2A(F'A' - 2FA'') + 4FA'^2 = 0, \tag{9}$$

$$-3A^2(\varepsilon H\Phi'^2 + 2F'') - 4A^2(\rho + q) + 6A(F'A' - 2FA'') + 12FA'^2 = 0, \tag{10}$$

$$H'\Phi'A + H(\Phi''A + 3\Phi'A') - \varepsilon N = 0. \tag{11}$$

The paper sets for hybrid model of $f(R,\Phi,X)$ function as follows (Bahamonde et al., 2015):

$$f(R,\Phi,X) = f_1(R) + f_2(\Phi,X). \tag{12}$$

By using Equations (8)-(11) and (12), one can get a solution set in the following form:

$$A(t) = C_1 t^{C_2}, \tag{13}$$

$$\Phi(t) = C_3 t^{-C_4}, \tag{14}$$

$$H(t) = -C_5 C_6 \Phi'^{2(C_6-1)}, \tag{15}$$

$$\rho(t) = -q, \tag{16}$$

where $C_i$ is arbitrary constants. The solution set is a model related with $f(R)$ theory or General Relative Theory with cosmological constant. $F(t)$ and $N(t)$ functions are obtained as:

$$F(t) = 8\left((C_4+1)^2 C_6^2 + \frac{(C_2-3)(C_4+1)C_6}{2} - C_2 + \frac{1}{2}\right) C_7 t^{\frac{C_2}{2}+\frac{1}{2}-\frac{\sqrt{C_2^2+10C_2+1}}{2}}$$

$$+8C_8\left((C_4+1)^2 C_6^2 + \frac{(C_2-3)(C_4+1)C_6}{2} - C_2 + \frac{1}{2}\right) t^{\frac{C_2}{2}+\frac{1}{2}+\frac{\sqrt{C_2^2+10C_2+1}}{2}}$$

$$+\varepsilon C_5 C_6 (-C_3 t^{-C_4-1} C_4)^{2C_6} t^2 / 8(C_4+1)^2 C_6^2 + 4(C_2 - 3)(C_4+1)C_6 - 8C_2 + 4, \tag{17}$$

$$N(t) = -\frac{1}{\varepsilon}((-1)^{2C_6} C_5 t^{(-2C_6+1)C_4-2C_6} C_3^{2C_6-1} C_4^{2C_6-1} C_6 (2C_6 C_4 - C_4 - 3C_2 + 2C_6 - 1). \quad (18)$$

If the necessary transformations are made and substituted into the equations, we can obtain our equations based on the Ricci Scalar:

$$F(R) = \left(\sqrt{6}\sqrt{\frac{C_2(2C_2-1)}{R}}\right)^{\frac{C_2}{2}+\frac{1}{2}+\frac{\sqrt{C_2^2+10C_2+1}}{2}} C_8(8C_4^2C_6^2 + 4C_4C_2C_6 + 16C_4C_6^2 - 12C_4C_6 + 4C_2C_6 + 8C_6^2 - 8C_2 - 12C_6 + 4) + \left(\sqrt{6}\sqrt{\frac{C_2(2C_2-1)}{R}}\right)^{\frac{C_2}{2}+\frac{1}{2}+\frac{\sqrt{C_2^2+10C_2+1}}{2}} C_7(8C_4^2C_6^2 + 4C_4C_2C_6 + 16C_4C_6^2 - 12C_4C_6 + 4C_2C_6 + 8C_6^2 - 8C_2 - 12C_6 + 4) + \frac{6\varepsilon C_5 C_6\left(-C_3\left(\sqrt{6}\sqrt{\frac{C_2(2C_2-1)}{R}}\right)^{-C_4-1} C_4\right)^{2C_6} C_2(2C_2-1)}{R} / 4(2C_4^2C_6^2 + C_4C_2C_6 + 4C_4C_6^2 - 3C_4C_6 + C_2C_6 + 2C_6^2 - 2C_2 - 3C_6 + 1), \quad (19)$$

$$N(R) = -\frac{1}{\varepsilon C_3 C_4}\left(C_5\left(\sqrt{6}\sqrt{\frac{C_2(2C_2-1)}{R}}\right)^{C_4}\right)\left(-C_3\left(\sqrt{6}\sqrt{\frac{C_2(2C_2-1)}{R}}\right)^{-C_4-1} C_4\right)^{2C_6} C_6(2C_4C_6 - C_4 - 3C_2 + 2C_6 - 1). \quad (20)$$

There were some dominant period transitions in the early universe. The last of these periods is the dark energy dominant period in today. We consider the periods of the universe:

(a) Radiation-Dominated Era

The simplest way to define this radiation-dominated era is to set the exponential expression of the time-dependent part of the metric potential function $A(t)$ equal to $1/2$, where the arbitrary constant $C_2$ is defined as indicated. In the case $\varepsilon = 1$ and $C_2 = 1/2$, $F(t) = 0$ and $N(t) = 0$. So, the selected model becomes a constant function and the solutin is reduced to the General Relativity theory.

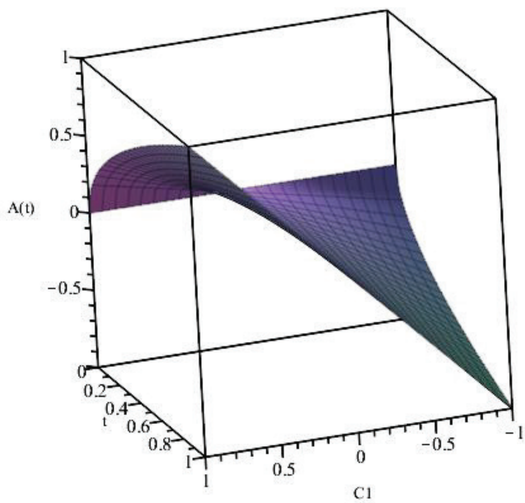


Fig.1. Scale factor $A(t)$ with $C_2 = \frac{1}{2}$, $0 < t < 1, -1 < C_1 < 1$.

Obtained scale factor, $A(t)$, given by Equation (13) is graphized in Figure (1). As can be seen, the $A(t)$ function is directly related to $C_1$. In the region where $C_1$ is negative, the metric potential takes positive values, while in the region where $C_1$ is positive, the metric potential decreases and takes negative values.

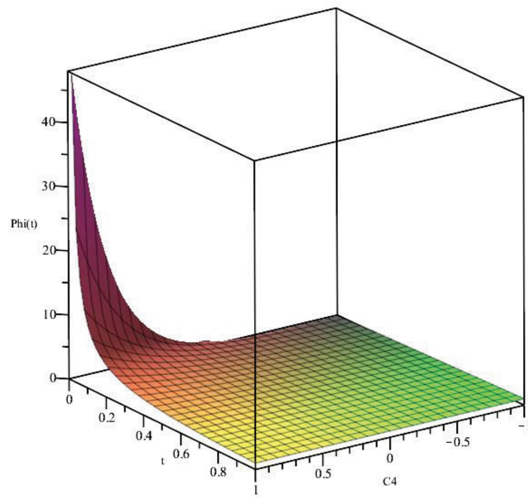


Fig.2. Scalar potential $\Phi(t)$ with $C_2 = \frac{1}{2}$, $C_3 = 1, 0 < t < 1, -1 < C_4 < 1$.

Figure (2) represents changes of the scalar potential given by Equation (14). As can be seen here, the scalar potential function, $\Phi(t)$, is always on the positive side. Over time, the scalar potential tends to decrease.

(b) Matter-Dominated Era

In order to define epoch in which matter-dominated era, same way as in the previous section, it is sufficient to make the exponential expression of the time-dependent part of the metric potential $A(t)$ function with $C_2 = 2/3$.

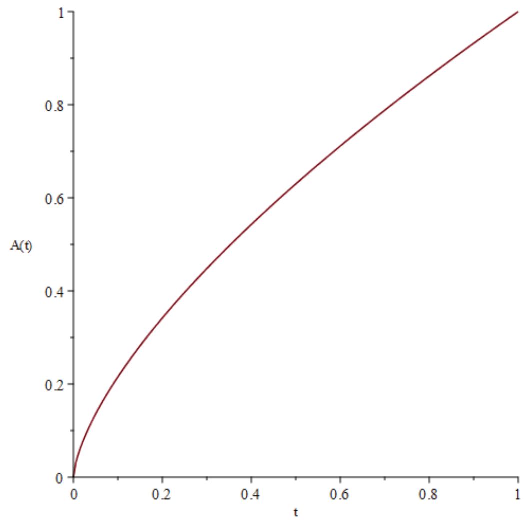


Fig.3. Scale factor $A(t)$ with $C_1 = 1, C_2 = \frac{2}{3}, 0 < t < 1$.

Figure (3) shows time-dependent variation of the metric potential under selected constants. The metric potential $A(t)$ shows an increasing behaviour in the positive part.

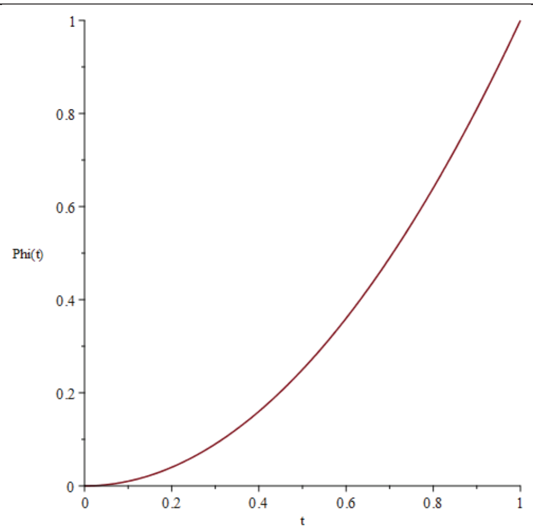


Fig.4. Scalar potential, $\Phi(t)$ with $C_3 = 1, C_4 = -2, 0 < t < 1$.

Similarly, Figure (4) shows us the time-dependent variation of the scalar potential under selected constants. The scalar potential $\Phi(t)$ shows an increasing behaviour in the positive part.

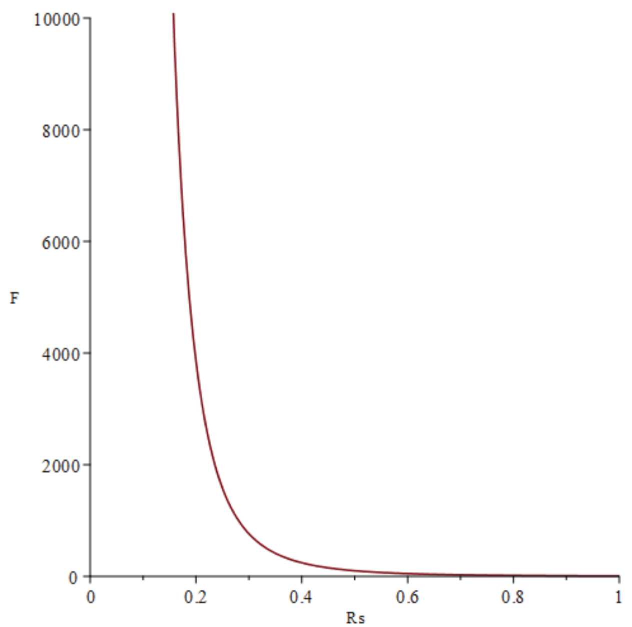


Fig.5. $F(R)$ function with $\varepsilon = 1$, $C_1 = C_3 = C_6 = 1$, $C_2 = \frac{2}{3}$, $C_3 = 1$, $C_5 = 0$.

Under the necessary transformations it is possible to translate the function $F(t)$ with respect to $t$ into a version that depends on the Ricci Scalar. Figure (5) shows us the behavior of the function $F(R)$ as regards the Ricci Scalar. Here we see that the function $F(R)$ gets positive values. In the same time it decreases rapidly for increasing values of the Ricci Scalar.

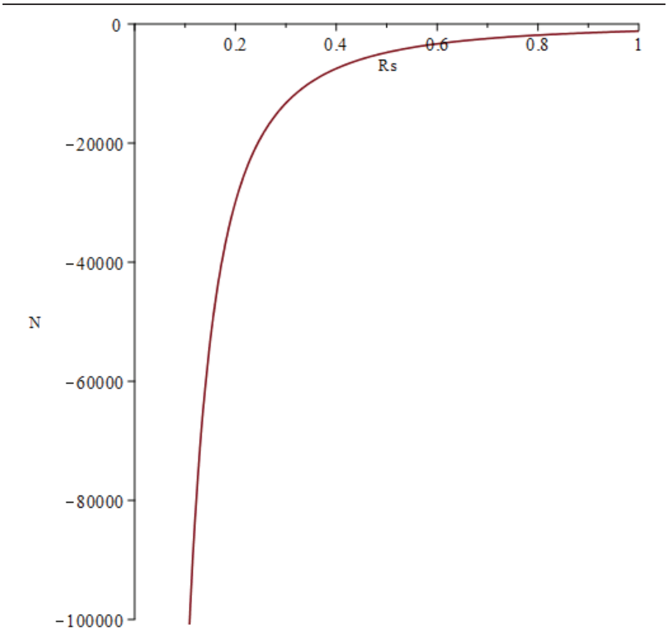


Fig.6. $N(R)$ function with $\varepsilon = 1, C_1 = C_3 = C_6 = 1, C_2 = \frac{2}{3}, C_3 = 1, C_5 = 0.$

Under the necessary transformations, it is possible to translate the function $N(t)$ depending on $t$ into a version depending on the Ricci Scalar. Figure (6) shows us the behavior of the function $N(R)$ depending on the Ricci Scalar. Here, the function $N(R)$ is not only on the negative side, but also increases rapidly for increasing values of the ricci scalar.

## 3. CONCLUSION

In this study, a string fluid filled universe, two dominant periods in the early universe are investigated within the scope of $f(R, \Phi, X)$ gravitation theory. $f(R, \Phi, X)$ gravitation theory includes a scalar potential and a kinetic term depending on this scalar potential, and the Ricci Scalar. The theory can be considered depending on a function instead of a constant, creates many advantages for the examination of the early periods of the universe. In the study, the field equations of the string fluid filled FRW universe and the Klein-Gordon equation are calculated with the selected theory. The $f(R, \Phi, X)$ model given by equation (12) is selected based on the obtained equations. Some cosmological findings are obtained in the context of this model. Then, the radiation dominant era and the matter dominant era of the universe are investigated with these findings. The graphs of the obtained results are drawn. The selected model in the radiation dominant era is reduced to Einstein's General Relativity (GR). The graphs of the $A(t)$ and $\Phi(t)$ functions are drawn in 3-dimensional form by determining arbitrary constants according to the parameters of these periods. The compatibility of the obtained graphs with Einstein's GR is observed.

$A(t)$ and $\Phi(t)$ functions are selected by choosing arbitrary constants suitable for the cosmic parameters for the model in the matter-dominant era. In addition, the necessary transformations are used and the $F(t)$ and $N(t)$ functions provided by the theory are given in terms of the Ricci Scalar in equations (19)-(20). The graphs of these equations are also given in figures (5)-(6). It is obvious that both graphs give the expected results.

In the study, the results show that $f(R, \Phi, X)$ theory is a consistent theory because it gives worth results related with early periods of universe such as radiation and/or matter dominated eras.

## 4. ACKNOWLEDGEMENT

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# Chapter 3

## NON-EXİSTENCE SOLUTIONS OF QUADRATIC EQUATIONS OF STATE IN SELF CREATION COSMOLOGY

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## 1. INTRODUCTION

Scientists have focused on studying the phenomenon following the late-time universe acceleration revealed by observations such as Supernova Ia (Perlmutter et al., 1999) cosmic microwave background radiation (CMB) (Gawiser & Silk, 2000), and Wilkinson Microwave Anisotropy Probe (Spergel, 2003). Because Einstein's Theory and the Mach principle were incomplete to explain this type of acceleration. The common view is that it would be possible to explain this acceleration by making use of alternative theories to Einstein General Relativity (GR) or dark matter such as k-essence, quintessence, and tachyon. In recent years, it has been observed that studies on some alternative theories such as Lyra theory (Lyra, 1951), Brans-Dicke theory (Brans and Dicke, 1961), Creation Field Cosmology (CFC) (Narlikar & Padmanabhan, 1985), *f(R,T)* theory (Harko, 2011) and Self Creation Cosmological (SCC) theory (Barber, 1982) etc. have increased. Reddy and Venkateswarlu (1988) have get nonexistence solutions of static conformally flat universe in the presence of source-free electromagnetic field. Venkateswarlu and Kumar (2006) have gotten solutions of Modified Field Equations (MEFEs) for higher dimensional FRW universe model with perfect fluid matter (PFM) by using an equation of state $p = \varepsilon\rho$, $(0 \leq \varepsilon \leq 1)$. Singh and Kumar (2007) have investigated Bianchi II universe filled with PFM in SCC theory. Rao et al. (2008) have studied GR and SCC theories in framework Bianchi II, VII and IX space-time for cosmic string matter. And Biachi type-V universe with PFM has been studied in SCC by Tiwari and Kumar (2010). Katore et al. (2010) have researched FRW space-time with bulk viscosity in SCC theory. Mahanta et al. (2012) have investigated Bianchi III universe with string quark matter in Barber's second theory. Chirde and Rahate (2012) have examined FRW cosmological solutions with massless scalar field and bulk viscous in framework SCC. Also, Jain and Jain (2015) have worked SCC for Bianchi type-I universe model with magnetized field. N-dimensional FRW universe model has been investigated in SCC with strange quark matter in form of string cloud by Şen and Aygün (2016). Çağlar and Aygün (2016a) have get solutions of SCC for n-dim flat FRW universe filled with domain wall (DW) and string cloud coupled quark matter forms. Hegazy and Rahaman (2019) have get non-existence solutions of Bianchi $VI_0$ space-time in SCC and GR theories. In this work quadratic equations of state (QEoS) that define the relationship between total pressure $p$ and energy density $\rho$ is given by have been used to get solutions of the constructed model. The first form of QEoS is written as follows.

$$p = p_0 + \alpha\rho + \beta\rho^2 \tag{1}$$

Here $p_0$, $\alpha$, and $\beta$ are constants (Ananda & Bruni, 2006). And the second form of QEoS named the restricted equation of state in the high energy regime is suggested as follows.

$$p = \alpha\rho + \frac{\rho^2}{\rho_c} \tag{2}$$

Where $\rho_c$ symbolizes characteristic energy scale (Ananda & Bruni, 2006). Also, the third model of the quadratic equations of state is written in following form (Singh & Bishi, 2015).

$$p = \alpha\rho^2 - \rho \tag{3}$$

Bianchi I space-time with PFM has been investigated in GR theory in the context of QEoS by Reddy et al. (2015). Singh and Bishi (2015) have studied in *f(R,T)* theory of the model. Şen and Aygün (2016) have get solutions of Lyra manifold with Bianchi type I universe filled with PFM by using QEoS. FRW universe with bulk viscosity and QEoS has been studied in GR theory by Singh et al. (2018). Aygün et al. (2019) have researched Marder universe model in the presence of PMF with QEoS form in *f(R,T)* gravity. Çağlar (2019) have investigated Bianchi V space-time with the QEoS form of PFM in *f(R,T)* theory including cosmological constant Λ.

It is well known that the late time universe is homogeneous and isotropic form, and it is described with FRW universe model. Aygün et al. (2015) have investigated n-dim FRW universe with quark matter in Lyra theory. Çağlar and Aygün (2016b) have get exact solutions of MEFEs for higher dimensional FRW universe model with strange quark matter (SQM) in Lyra theory. Also, Çağlar and Aygün (2016c) have put forward non-existence solutions of n-dimensional FRW space-time in Brans-Dicke theory with SQM and DW. *(n+2)*-dim FRW universe model with SQM has been studied in CFC theory by Aygün et al. (2016). Goyal et al. (2019) have examined the homogeneous and isotropic FRW universe for PFM in GR theory with time variable cosmological constant Λ. Tiwari et al. (2021) have researched FRW universe in *f(R,T)* theory with PFM. Chaudhary et al. (2023) have investigated *f(R,G,T)* theory with Λ for FRW space-time. Shukla et al. (2024) have worked FRW universe in *f(Q)* theory with a parameterization of the EoS parameter.

In this study, higher dimensional FRW universe model with PFM in form of QEoS has been investigated in CFC theory by taking the above-mentioned studies as examples. Firstly, the cosmological model has been constructed, and then the solutions of MEFEs have been obtained by using QEoS. At the end of the study, all solutions have been concluded in the last section.

## 2. MODIFIED FIELD EQUATIONS OF SCC

Self Creation Cosmological theory which is one of Barber's theories is described by following equation.

$$R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi}{\Phi} T_{ik} \tag{4}$$

Where $T_{ik}$ represents energy-momentum tensor of matter distribution (Barber, 1982). And $\Phi$ symbolizes the scalar field that satisfies the equation given in the following form.

$$\Box\, \Phi = \Phi^{k}_{;k} = \frac{8\pi}{3} \lambda T_{ik} \tag{5}$$

Here coupling constant given $\lambda$ takes value as $\lambda < 0.1$ depending on experimental measurements (Katore & Shaikh, 2011). The solutions of Barber's second theory reduce to Einstein's General Relativity solutions when $\Phi = constant$. The perfect fluid matter has been assumed as the matter source of the universe and it can be defined as follows.

$$T_{ik} = (\rho + p) u_i u_k - p g_{ik} \tag{6}$$

and

$$T^{ik}_{;k} = 0 \tag{7}$$

Where $\rho$ and $p$ respectively symbolize energy density and total pressure of perfect fluid matter (Dalal, 2024). $g_{ik}$ is a metric tensor, and $u_i$ is named dimensional velocity $u_i = (1, 0, 0, \ldots 0)$. The $(n+2)$ dimensional flat FRW universe line element is written as follows.

$$ds^2 = dt^2 - R(t)^2[dr^2 + r^2 d\chi_n^2] \tag{8}$$

Here $R$ is the scale factor of the universe model and $d\chi_n^2$ is the short form of the form whose explicit expression is given below (Singh & Beesham, 2012).

$$d\chi_n^2 = d\theta_1^2 + \sin^2\theta_1\, d\theta_2^2 + \sin^2\theta_1\ \sin^2\theta_2 + \cdots + \sin^2\theta_{n-1}\, d\theta_n^2 \tag{9}$$

On the other hand, the Hubbel parameter and deceleration parameter are given for the higher dimensional flat FRW universe as follows (Godani, 2019).

$$H = \frac{\dot{V}}{(n+1)V} = \frac{\dot{R}}{R} \tag{10}$$

$$q = \frac{\partial}{\partial t}\left(\frac{1}{H}\right) - 1 = \frac{R\ddot{R}}{\dot{R}} \tag{11}$$

Here and after, the upper dot ( . ) represents differentiation with respect to cosmic time and $V$ symbolizes $V = \sqrt{\det g_{ik}}$. The modified field equations of the higher dimensional flat FRW universe filled with perfect fluid matter for Self Creation Cosmology have been attained as follows by using eqs. (3), (5) and (7).

$$n\frac{\ddot{R}}{R} + \frac{n(n-1)}{2}\frac{\dot{R}^2}{R^2} = -\frac{8\,\pi\,p}{\Phi} \tag{12}$$

$$\frac{n(n+1)}{2}\frac{\dot{R}^2}{R^2} = \frac{8\,\pi\,\rho}{\Phi} \tag{13}$$

Also, from eq. (7) it can be attained as follows.

$$(n+1)(p+\rho)\frac{\dot{R}}{R} + \dot{\rho} = 0 \tag{14}$$

## 3. SOLUTIONS OF THE MEFEs

Now, there are three equations (12)-(14) for four unknown quantities like $R$, $\rho$, $p$ and $\Phi$. Thus, one assumption has been needed to solve MEFEs. In this letter quadratic equations of state (QEoS) have been used to attain values of the four unknown quantities and they are given as following subsection.

### 3.1. Solutions for the First Model of QEoS

In this work, firstly the eq. (1) has been used with eqs. (12)-(14) and the scale factor $R$, energy density $\rho$, and scalar field of SCC $\Phi$ are calculated as follows, respectively.

$$R_1 = s_2 e^{s_1 t} \tag{15}$$

$$\rho_1 = \frac{1}{2\beta}\left[\sqrt{\alpha^2 + 2\alpha + 1 - 4\beta p_0} - \alpha - 1\right] \tag{16}$$

$$\Phi_1 = \frac{8\pi}{n(n+1)\beta s_1^2}\left[\sqrt{\alpha^2 + 2\alpha + 1 - 4\beta p_0} - \alpha - 1\right] \tag{17}$$

And total pressure $p$ of the model has been calculated from eqs. (1) and (16) as follows.

$$p_1 = \frac{1}{2\beta}\left[\alpha + 1 - \sqrt{\alpha^2 + 2\alpha + 1 - 4\beta p_0}\right] \tag{18}$$

Also, the Hubble parameter and deceleration parameter have been obtained by using eq. (15) in eqs. (10) and (11) as following forms.

$$H_1 = s_1 \tag{19}$$

$$q_1 = -s_1 s_2 e^{s_1 t} \tag{20}$$

3.2. Solutions for the Second Model of QEoS

When the eq. (2) has been used with eqs. (12)-(14) and the scale factor $R$, energy density $\rho$, and scalar field of SCC $\Phi$ are calculated as follows, respectively.

$$R_2 = c_1 \tag{21}$$

$$\rho_2 = 0 \tag{22}$$

$$\Phi_2 = c_2\, t + c_3 \tag{23}$$

And total pressure $p$ of the model has been calculated from eqs. (2) and (22) as follows.

$$p_2 = 0 \tag{24}$$

Also, Hubble parameter has been obtained by using eq. (19) in eq (10) as following forms.

$$H_2 = 0 \tag{25}$$

3.3. Solutions for the Second Model of QEoS

When the eq. (3) has been used with eqs. (12)-(14) and the scale factor $R$, energy density $\rho$, and scalar field of SCC $\Phi$ are calculated as follows, respectively.

$$R_3 = m_1 \tag{26}$$

$$\rho_3 = 0 \tag{27}$$

$$\Phi_3 = m_2\, t + m_3 \tag{28}$$

And total pressure $p$ of the model has been calculated from eqs. (3) and (27) as follows.

$$p_3 = 0 \tag{29}$$

Also, the Hubble parameter has been obtained by using eq. (26) in eq. (10) as following form.

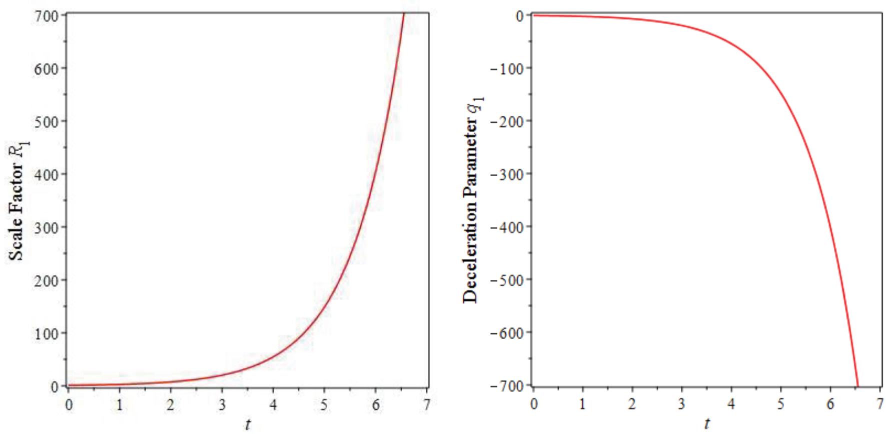
$$H_3 = 0 \tag{30}$$

## 4. CONCLUSION

In this study, the behavior of SCC theory, one of the alternative gravitation theories, with perfect fluid matter was examined for higher dimensional flat FRW universe model. The quadratic equations of state (QEoS), given by eqs. (1) - (3), have been used to solve MEFEs. It is attained that the ($n$+2)-dim flat FRW space-time filled with perfect fluid matter does not allow Self Creation Cosmology (SCC) solutions due to obtaining constant scalar field $\Phi$ for two QEoS models. The new line elements can be rewritten for the first model of QEoS from eq. (8) with eq. (15) as follows.

$$ds_1^2 = dt^2 - s_2^2 e^{2s_1 t}\,[dr^2 + r^2 d\chi_n^2] \tag{31}$$

Constant parameters $s_1$ and $s_2$ must be different from zero. It is attained that the first model of the universe starts at $s_1 \rightarrow 0$ and ends at $s_2 \rightarrow 0$. And, it can be said that the perfect fluid matter of the first model plays a role in dark matter due to the attained $p_1 = -\rho_1$ from eqs. (16) and (18) (Babichev, 2004). Also, eqs. (15) and (20) give the accelerating expanding universe for the first QEoS solutions of the constructed model and it is shown in Fig 1. Thus, it can be concluded that the perfect fluid matter may cause this expansion for constructed first model of QEoS.




a) Scale Factor $R_1$

b) Deceleration parameter $q_1$

Fig.1. Variations of $R_1$ and $q_1$ concerning cosmic time by assuming $s_1 = s_2 = 1$.

On the other hand, except for the scalar field, other quantities are obtained as dimensionless. Thus when $n = 2$, one can get a solution of scalar field $\Phi$ for 4-dim (4D) flat FRW universe as follows.

$$\Phi_1^{4D} = \frac{4\pi}{3\beta s_1^2}\left[\sqrt{\alpha^2 + 2\alpha + 1 - 4\beta p_0} - \alpha - 1\right] \tag{32}$$

The new line elements can be rewritten for the second and third model of QEoS from eq. (8) with eqs. (21) and (26) respectively as follows.

$$ds_2^2 = dt^2 - c_1^2\,[dr^2 + r^2 d\chi_n^2] \tag{33}$$

$$ds_3^2 = dt^2 - m_1^2\,[dr^2 + r^2 d\chi_n^2] \tag{34}$$

It is clear to see that the new line element of the constructed model is time independent for second and third models QEoS. Also, the solutions turn into vacuum solutions when the second and third models of QEoS are assumed for constructed model. If arbitrary constants $c_2$ and $m_2$ in eqs. (23) and (28) are assumed equal to zero, obtained solutions reduce to GR solutions. On the other hand, Hubble parameters of these two QEoS models have been calculated as zero, and this means the constructed models are static universe type. In general, it is possible to say that solutions of MEFEs for higher dimensional FRW universe with PFM do not survive in SCC for second and third QEoS.

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# Chapter 4

## BIANCHI-I UNIVERSE WITH MASSLESS SCALAR FIELD IN UNIMODULAR F(R,T) GRAVITY

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## 1. INTRODUCTION

Modified theories are introduced in order to address cosmological problems such as cosmological constant problem, dark energy, dark matter etc. Unimodular $f(R,T)$ gravity is recently developed modified gravitational theory. $f(R,T)$ function is substituted with curvature scalar (R) in Einstein-Hilbert action. Here, energy-momentum tensor's trace is shown by T. Additionally, action function contains a Lagrange multiplier with unimodular constraint. Unimodularity condition is given by $\sqrt{-g} = \epsilon$, where $\epsilon$ is any constant (Eichhorn, 2015). Fixed determinant of the metric can be satisfied with a suitable coordinate transformation. Unimodular $f(R,T)$ gravity was developed by Rajabi and Nozari (2017) in Jordan and Einstein frames. They obtained that Lagrange multiplier varies over time in Jordan frame and equivalent to cosmological constant in Einstein frame.

Singh, Bishi and Sahoo (2016) researched Bianchi-I spacetime filled with scalar field in $f(R,T)$ gravity by using variable cosmological constant. Standard $f(R,T)$ gravity is examined in Bianchi-I spacetime with bulk viscosity by P. K. Sahoo, P. Sahoo and Bishi (2017). They used a time varying deceleration parameter with two different $f(R,T)$ models. Kanakavalli and Ananda-Rao (2016) studied standard $f(R,T)$ gravity in Bianchi-I spacetime with Nambu and Takabayasi strings. Sharif and Zubair (2012) researched anisotropic Bianchi-I spacetime by using massless scalar field and perfect fluid in standard $f(R,T)$ gravity. They take into account kinematical quantities including mean Hubble parameter and energy conditions with various cosmological models. Tiwari, Sofuoğlu and Dubey (2020) examined Bianchi-I spacetime with a particular deceleration parameter in standard $f(R,T)$ gravity. Houndjo (2017) researched Bianchi-I and Friedmann-Robertson-Walker metric in unimodular $f(G)$ gravity by using de Sitter and power law models. Taşer and Ulu Doğru (2020) obtained solutions for each of massive and massless scalar field in Bianchi-I spacetime in $f(G)$ gravity context. They found that massless scalar field acts as quintessence for some rebuilded models. More studies on Bianchi-I spacetime in modified gravitational theories can be found in Refs. (Sharif and Rani, 2011; Adhav, 2012; Çağlar, Aygün, Aktaş and Taşer, 2014; Singh and Bishi, 2015; Tiwari, Beesham and

Shukla, 2018; Afonso, Olmo, Orazi and Rubiera-Garcia, 2019; De Cesare and Wilson-Ewing, 2019).

In this paper, we study energy conditions, physical and kinematical quantities in unimodular $f(R,T)$ gravity by choosing anisotropic Bianchi-I spacetime and massless scalar field matter source. In section 2, unimodular Bianchi-I spacetime is defined and the field equation solutions are achieved. Section 3 is allocated for energy conditions and kinematical quantities. Section 4 contains concluding commentations.

## 2. UNIMODULAR $f(R,T)$ GRAVITY AND MASSLESS SCALAR FIELD IN UNIMODULAR BIANCHI-I SPACETIME

Action function of unimodular $f(R,T)$ gravity is represented with

$$S = \frac{1}{2}\int d^4x\left[\sqrt{-g}\, f(R,T) - 2\lambda\left(\sqrt{-g} - \epsilon\right)\right] + \int d^4x\sqrt{-g}\, L_m. \quad (1)$$

Here $L_m$ denotes matter Lagrangian. $\epsilon = 1$ is assumed (Rajabi and Nozari, 2017). By using the least action principle $\delta S = 0$, the field equation yields

$$f_{,R}(R,T)\, R_{ik} - \frac{1}{2} g_{ik} f(R,T) + (g_{ik}\,\Box - \nabla_i \nabla_k) f_{,R}(R,T) + \lambda g_{ik} - T_{ik} + f_{,T}(R,T)(T_{ik} + \Theta_{ik}) = 0 \quad (2)$$

where $\Theta_{ik}$ is described with

$$\Theta_{ik} = g^{ab}\frac{\delta T_{ab}}{\delta g^{ik}} = -2T_{ik} + g_{ik} L_m. \quad (3)$$

Here, $\Box$ and $\nabla$ denote d'Alembert operator and covariant derivative, respectively (Rajabi and Nozari, 2017). $\frac{\delta}{\delta g^{ik}}$ shows variation with respect to metric tensor. Also, $f_{,R}$ and $f_{,T}$ represent $f_{,R}(R,T) = \frac{df(R,T)}{dR}$ and $f_{,T}(R,T) = \frac{df(R,T)}{dT}$, respectively. $\Theta_{ik}$ for massless scalar field is

$$\Theta_{ik} = -2T_{ik} - \frac{1}{2} g_{ik}\left(\Phi^{,l}\Phi_{,l}\right). \tag{4}$$

Trace of Eq. (2) is obtained as

$$f_{,R}(R,T)R + 3\Box f_{,R}(R,T) - 2f(R,T) + \lambda - T + f_{,T}(R,T)(T+\Theta) = 0 \tag{5}$$

where $T = g^{ik}T_{ik}$ and $\Theta = g^{ik}\Theta_{ik}$. Substituting Eq. (5) into Eq. (2) yields

$$f_{,R}(R,T)\left[R_{ik} - \frac{1}{3}g_{ik}R\right] - \nabla_i\nabla_k f_{,R}(R,T) + \frac{1}{6}f(R,T)g_{ik} - \frac{1}{3}\lambda g_{ik} - \left[T_{ik} - \frac{1}{3}g_{ik}T\right] + f_{,T}(R,T)\left[T_{ik} - \frac{1}{3}g_{ik}T\right] + f_{,T}(R,T)\left[\Theta_{ik} - \frac{1}{3}g_{ik}\Theta\right] = 0 \tag{6}$$

In unimodular $f(R,T)$ gravity, the line element must be transformed to unimodular form. A cosmic time coordinate transformation can be used to ensure unimodular condition. Bianchi-I spacetime metric can be denoted as

$$ds^2 = A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2 - dt^2. \tag{7}$$

Here, $A(t)$, $B(t)$, $C(t)$ indicate scale factors. A cosmic time $(\tau)$ coordinate transformation such as $d\tau = A(t)B(t)C(t)dt$ satisfies unimodularity (Houndjo, 2017). Substituting $d\tau$ into Eq. (7) gives

$$ds^2 = A^2(\tau)dx^2 + B^2(\tau)dy^2 + C^2(\tau)dz^2 - \frac{1}{A^2(\tau)B^2(\tau)C^2(\tau)}d\tau^2 \tag{8}$$

Energy-momentum tensor (EMT) for massless scalar field is given by

$$T_{ik} = \Phi_{,i}\Phi_{,k} - \frac{1}{2} g_{ik}\left(g^{ab}\Phi_{,a}\Phi_{,b}\right) \tag{9}$$

where $\Phi(\tau)$ denotes scalar potential (Hawking and Ellis, 1973). Components of EMT for scalar field and its trace are obtained as, respectively,

$$T_i^k = diag\left[\frac{(ABC)^2(\dot{\Phi})^2}{2}, \frac{(ABC)^2(\dot{\Phi})^2}{2}, \frac{(ABC)^2(\dot{\Phi})^2}{2}, -\frac{(ABC)^2(\dot{\Phi})^2}{2}\right] \tag{10}$$

$$T = (ABC)^2\left(\dot{\Phi}\right)^2 \tag{11}$$

Also, $\Theta_i^k$ components and its trace yields

$$\Theta_i^k = diag\left[-\frac{(ABC)^2(\dot{\Phi})^2}{2}, -\frac{(ABC)^2(\dot{\Phi})^2}{2}, -\frac{(ABC)^2(\dot{\Phi})^2}{2}, 3\frac{(ABC)^2(\dot{\Phi})^2}{2}\right], \tag{12}$$

$$\Theta = 0. \tag{13}$$

We use $f(R,T) = R + 2f_2(T)$ function among viable models. In this context, the field equations are obtained as

$$\ddot{f}_R - f_R\left(\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2}\right) + \dot{f}_R\frac{\dot{A}}{A} - 2f_R\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC}\right) + 2\dot{f}_R\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{\dot{\Phi}^2}{2} - \frac{1}{(ABC)^2}\left(\lambda - \frac{f}{2}\right) = 0 \tag{14}$$

$$\ddot{f}_R - f_R\left(\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2}\right) + \dot{f}_R\frac{\dot{B}}{B} - 2f_R\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC}\right) + 2\dot{f}_R\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right) + \frac{\dot{\Phi}^2}{2} - \frac{1}{(ABC)^2}\left(\lambda - \frac{f}{2}\right) = 0 \tag{15}$$

$$\ddot{f}_R - f_R\left(\frac{\ddot{C}}{C} + \frac{\dot{C}^2}{C^2}\right) + \dot{f}_R\frac{\dot{C}}{C} - 2f_R\left(\frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right) + 2\dot{f}_R\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \frac{\dot{\Phi}^2}{2} - \frac{1}{(ABC)^2}\left(\lambda - \frac{f}{2}\right) = 0 \tag{16}$$

$$f_R\left(\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{A}^2}{A^2}+\frac{\dot{B}^2}{B^2}+\frac{\dot{C}^2}{C^2}+\frac{2\dot{A}\dot{B}}{AB}+\frac{2\dot{A}\dot{C}}{AC}+\frac{2\dot{B}\dot{C}}{BC}\right)-\dot{f}_R\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)+ \left(f_T+\frac{1}{2}\right)\frac{\dot{\Phi}^2}{2}+\frac{1}{(ABC)^2}\left(\lambda-\frac{f}{2}\right)=0 \quad (17)$$

Also, Klein-Gordon equation is given by

$$\nabla^i\nabla_i\Phi=0 \quad (18)$$

For $f(R,T) = R + 2f_2(T)$ model, Eqs. (14)-(18) can be solved together as follows:

$$A(\tau)=-\frac{c_3c_4e^{c_3c_7(\tau+c_6)}}{c_5} \quad (19)$$

$$B(\tau)=-\frac{c_2c_5}{c_3c_4e^{c_3c_7(\tau+c_6)}} \quad (20)$$

$$C(\tau)=c_1\sqrt{-c_5(\tau+c_6)} \quad (21)$$

$$\Phi(\tau)=-\frac{\ln(\tau+c_6)}{c_5}+c_8 \quad (22)$$

Here $c_i$ ($i$ = 1, 2, 3, ...) denotes integration constants. Real values of scalar potential $\Phi(\tau)$ can be ensured with $(\tau+c_6)>0$. So, for real and non-vanishing $C(\tau)$ metric potential, it must be $c_5<0$. The solution also gives

$$f_{,T}=-2c_5^2-1 \quad (23)$$

$$f(\tau)=\frac{(c_1c_2)^2}{c_5(\tau+c_6)}+2\lambda \quad (24)$$

Curvature scalar and trace of EMT yield

$$R = -\frac{2c_5(c_1c_2)^2}{\tau+c_6} \tag{25}$$

$$T = -\frac{(c_1c_2)^2}{c_5(\tau+c_6)} \tag{26}$$

Substituting $\tau(T)$ function into $f(R,T)$ gives

$$f(R,T) = R - \left(2c_5^2 + 1\right)T + 2\lambda \tag{27}$$

The result in Eq. (27) implies $f(R,T) = R + \mu T$ type $f(R,T)$ model where μ is any constant. Here $\mu$ can be evaluated as effective cosmological constant (Harko, Lobo, Nojiri and Odintsov, 2011). For our solution, $\mu$ is being a negative constant. If we substitute Eq. (27) into Eq. (2), $\lambda$ term vanishes.

$$R_{ik} - \frac{1}{2}g_{ik}R - \frac{1}{2}g_{ik}(R + \mu T) - T_{ik} + \mu(T_{ik} + \Theta_{ik}) = 0 \tag{28}$$

In this case, $\mu$ can play an effective cosmological constant role. Negative cosmological constant causes to contraction of spacetime. Such spacetimes also called anti-de Sitter (AdS) spacetimes. Stiff matter has positive pressure with $p = \rho$, so it has contracting behavior.

## 3. ENERGY CONDITIONS AND KINEMATICAL QUANTITIES

At this chapter, strong (SEC) and weak (WEC) energy conditions will be presented in unimodular $f(R,T)$ gravity for Bianchi-I spacetime. We can take into account Raychaudhuri equations for descriptions of energy conditions (Albareti, Cembranos, de la Cruz-Dombriz and Dobado, 2014). Raychaudhuri equations are given as follows (Santos, Alcaniz, Reboucas and Carvalho, 2007).

$$\frac{d\vartheta}{d\Im} = -\frac{1}{3}\,\vartheta^2 + \omega_{ik}\omega^{ik} - \sigma_{ik}\sigma^{ik} - R_{ik}\,u^i u^k \tag{29}$$

$$\frac{d\vartheta}{d\Im} = -\frac{1}{2}\,\vartheta^2 + \omega_{ik}\omega^{ik} - \sigma_{ik}\sigma^{ik} - R_{ik}n^i\,n^k \tag{30}$$

Here $\omega_{ik}$ denotes rotation tensor and $\sigma_{ik}$ is being shear tensor. Additionally, $u^i$ and $n^i$ represent time-like and light-like vectors, respectively. Light-like (null) vectors satisfy $g_{ik}n^i\,n^k = 0$, so $n^i = \left[\frac{1}{A^2}, 0,0, BC\right]$. Also, four-vectors are $u^i = [0,0,0, ABC]$. Expansion scalar can be denoted as $\vartheta = -\Im R_{ik}\,u^i u^k$ and $\vartheta = -\Im R_{ik}\,n^i\,n^k$, respectively (Santos, Alcaniz, Reboucas and Carvalho, 2007). Attractive gravity requires $\vartheta < 0$, therefore it must be satisfied $R_{ik}\,u^i u^k \geq 0$ and $R_{ik}\,n^i\,n^k \geq 0$. The inequalities are expressed with effective EMT $\left(T_{ik}^{eff}\right)$ as:

$$\begin{aligned} &\left(T_{ik}^{eff} - \tfrac{1}{2}g_{ik}T^{eff}\right)u^i u^k \geq 0, \\ &\left(T_{ik}^{eff} - \tfrac{1}{2}g_{ik}T^{eff}\right)n^i n^k \geq 0, \\ &T_{ik}^{eff}u^i u^k \geq 0. \end{aligned} \tag{31}$$

where $T^{eff}$ is the trace of effective EMT. By using Eq. (2), $T^{eff}$ and $T_{ik}^{eff}$ for unimodular $f(R,T)$ gravity yield

$$T_{ik}^{eff} = \frac{1}{f_{,R}}\left[T_{ik} - f_{,T}(T_{ik} + \Theta_{ik}) + \frac{1}{2} g_{ik}\left(f - 2\lambda - Rf_{,R}\right) - (g_{ik}\,\Box - \nabla_i\nabla_k) f_{,R}\right], \tag{32}$$

$$T^{eff} = \frac{1}{f_{,R}}\left[T - f_{,T}(T + \Theta) + 2\left(f - 2\lambda - Rf_{,R}\right) - 3\Box f_{,R}\right]. \tag{33}$$

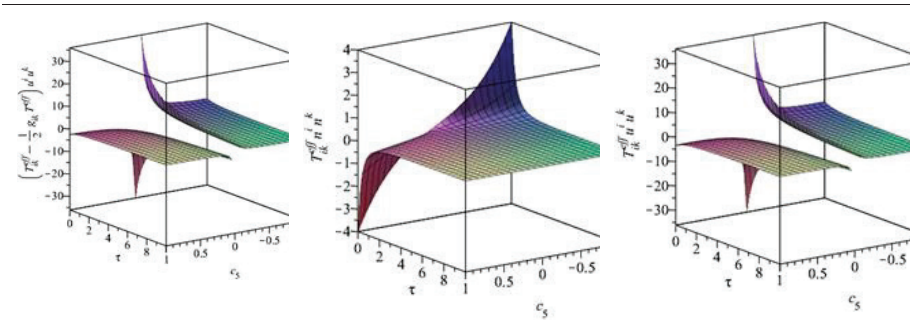
By using Eqs. (31)-(33), we get SEC and WEC as follows:

$$-\frac{(c_1c_2)^2(2c_5^2+3)}{2c_5(\tau+c_6)} \geq 0, \tag{34}$$

$$-\frac{2c_5(c_1c_2)^2(c_5^2+1)e^{-2c_3c_7}}{(c_3c_4)^2(\tau+c_6)^3} \geq 0, \tag{35}$$

$$-\frac{(c_1c_2)^2(4c_5^2+3)}{2c_5(\tau+c_6)} \geq 0. \tag{36}$$

Here Eqs. (34) and (35) ensure SEC. Also, Eqs. (35) and (36) are WEC while Eq. (35) represents null energy condition (NEC). It can be seen that SEC and WEC are satisfied when $c_5 < 0$. The conditions in Eqs. (34)-(36) are plotted in Fig. 1.




(a) (b) (c)

Fig. 1. Graphs of SEC and WEC with respect to time ($\tau$) for $c_1 = c_2 = c_3 = c_4 = 1, -1 \leq c_5 \leq 1, c_6 = 1, c_7 = 0$, (a) Eq. (34), (b) Eq. (35), (c) Eq. (36).

Mean Hubble parameter differs for unimodular Bianchi-I spacetime.

$$\mathcal{H}(\tau) = \frac{1}{3}\frac{d\tau}{dt}\left(\frac{\frac{dA}{d\tau}}{A} + \frac{\frac{dB}{d\tau}}{B} + \frac{\frac{dC}{d\tau}}{C}\right) = \frac{1}{3}\frac{d}{d\tau}(ABC) \quad (37)$$

It is recalled $d\tau = A(t)B(t)C(t)dt$ transformation in above equation.

$$\mathcal{H}(\tau) = -\frac{c_1 c_2 c_5}{6\sqrt{-c_5(\tau + c_6)}} \quad (38)$$

For positive mean Hubble parameter $\mathcal{H}(\tau)$, $c_1$ and $c_2$ must be $c_1 c_2 > 0$ and $c_5 < 0$. 4- velocities are taken into account for definition of kinematical quantities. Kinematical quantities can be obtained as follows (Culetu, 1994):

Expansion scalar yields,

$$\tilde{\vartheta}(\tau) = \nabla_i u^i = -\frac{c_1 c_2 c_5}{2\sqrt{-c_5(\tau + c_6)}} = 3\mathcal{H}(\tau) \quad (39)$$

4-acceleration shows whether non-gravitational force exist (Culetu, 1994). 4-acceleration in co- moving frame can be obtained as

$$\alpha^i = u^k \nabla_k u^i = (0,0,0,0). \quad (40)$$

Shear tensor is given by,

$$\tilde{\sigma}_{ik} = \frac{1}{2}\left(h_k^l \nabla_l u_i + h_i^l \nabla_l u_k\right) - \frac{1}{3}\vartheta\, h_{ik} + \frac{1}{2}(\alpha_i u_k + \alpha_k u_i) \quad (41)$$

where $h_{ik} = g_{ik} + u_i u_k$. Shear scalar is,

$$\tilde{\sigma}^2(\tau) = \frac{1}{2}\tilde{\sigma}_{ik}\tilde{\sigma}^{ik} = -\frac{13}{12}\frac{c_5(c_1c_2)^2}{(\tau+c_6)}. \tag{42}$$

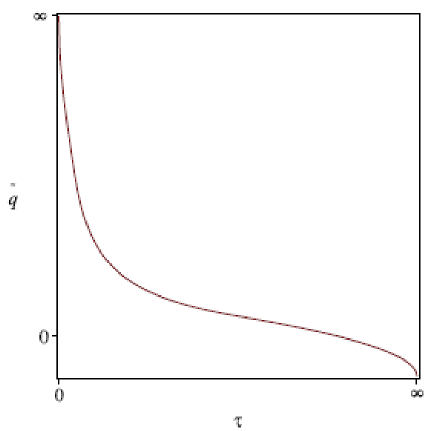
The ratio of shear and expansion scalar yields

$$\frac{\tilde{\sigma}^2(\tau)}{\tilde{\vartheta}^2(\tau)} = \frac{13}{3}. \tag{43}$$

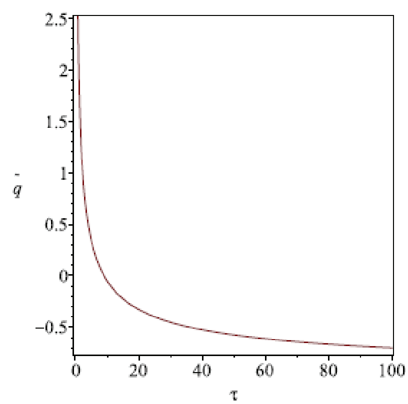
Eq. (43) implies that the spacetime always remains anisotropic in unimodular coordinates. Deceleration parameter in unimodular coordinates can be obtained by using $\mathcal{H}(\tau)$,

$$\tilde{q}(\tau) = -\frac{\dot{\mathcal{H}}(\tau)}{\mathcal{H}^2(\tau)} - 1 = \frac{3}{c_1c_2\sqrt{-c_5(\tau+c_6)}} - 1. \tag{44}$$

As it can be seen, $\tilde{q}$ parameter depends on $\tau$. Graphics of the $\tilde{q}(\tau)$ are plotted in Fig. 2.




(a)




(b)

Fig. 2. Deceleration parameter ($\tilde{q}$) with respect to time ($\tau$) for $c_1 = c_2 = 1$, , $c_5 = -1$,
$c_6 = 0$, (a)$0 \leq \tau < \infty$, (b) $0 \leq \tau \leq 100$.

Anisotropy parameter with $\mathcal{H}(\tau)$ is obtained as

$$\widetilde{\triangle} = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\mathcal{H}_i - \mathcal{H}}{\mathcal{H}}\right)^2 = \frac{2c_1c_2\sqrt{-c_5(\tau+c_6)} - 27}{(c_1c_2)^2 c_5(\tau+c_6)} + 1. \tag{45}$$

where $H_i$ represents directional Hubble parameter such as $H_x, H_y, H_z$. While $\tau \to \infty$, anisotropy parameter $\widetilde{\triangle}$ tends to a constant value.

At this point, we can apply reverse coordinate transformation:

$$dt = \frac{d\tau}{A(\tau)B(\tau)C(\tau)} = \frac{d\tau}{c_1c_2\sqrt{-c_5(\tau+c_6)}}. \tag{46}$$

We obtain $t(\tau)$ function as follows:

$$t(\tau) = \frac{2(\tau+c_6)}{c_1c_2\sqrt{-c_5(\tau+c_6)}}. \tag{47}$$

Also, $\tau(t)$ function is calculated as

$$\tau(t) = -c_5\left(\frac{c_1c_2}{2}\right)^2 t^2 - c_6. \tag{48}$$

Substituting $\tau(t)$ into Eqs. (19)-(21) gives

$$A(t) = c_3c_4e^{c_3c_7}\left(\frac{c_1c_2}{2}\right)^2 t^2\ , \tag{49}$$

$$B(t) = \frac{4}{c_1^2 c_2 c_3 c_4 e^{c_3 c_7} t^2}, \tag{50}$$

$$C(t) = \frac{1}{2} c_1^2 c_2 c_5 t \, . \tag{51}$$

By using Eqs. (49)-(51) Hubble parameter and kinematical quantities can be recalculated. Average Hubble parameter H(t) is obtained as

$$H(t) = \frac{1}{3}\left(\frac{\frac{dA(t)}{dt}}{A(t)} + \frac{\frac{dB(t)}{dt}}{B(t)} + \frac{\frac{dC(t)}{dt}}{C(t)}\right) = \frac{1}{3t} \, . \tag{52}$$

By using H(t), deceleration parameter $q$ yields

$$q(t) = -\frac{\dot{H}(t)}{H^2(t)} - 1 = 2. \tag{53}$$

This result corresponds to decelerating expansion of the Universe which implies matter dominated era. Same result $(q = 2)$ was obtained by Shamir (Shamir, 2015) in standard $f(R, T)$ gravity by using Bianchi-I metric with perfect fluid. By using H(t), anisotropy parameter is found as

$$\triangle = 26. \tag{54}$$

By using $t$ coordinate, expansion scalar, shear scalar and their ratio can be written respectively as follows:

$$\vartheta = \frac{1}{t}, \tag{55}$$

$$\sigma^2 = \frac{13}{3t^2}, \tag{56}$$

$$\frac{\sigma^2}{\vartheta^2} = \frac{13}{3}. \tag{57}$$

These results also show that anisotropy is continuous. Mean Hubble parameter corresponds stiff matter ($p = \rho$) dominated universe. Scalar field may be translated other matter forms by using density ($\rho$) and pressure ($p$) (Madsen, 1988).

$$\rho = \frac{(\Phi')^2}{2} + V(\Phi), \tag{58}$$

$$p = \frac{(\Phi')^2}{2} - V(\Phi). \tag{59}$$

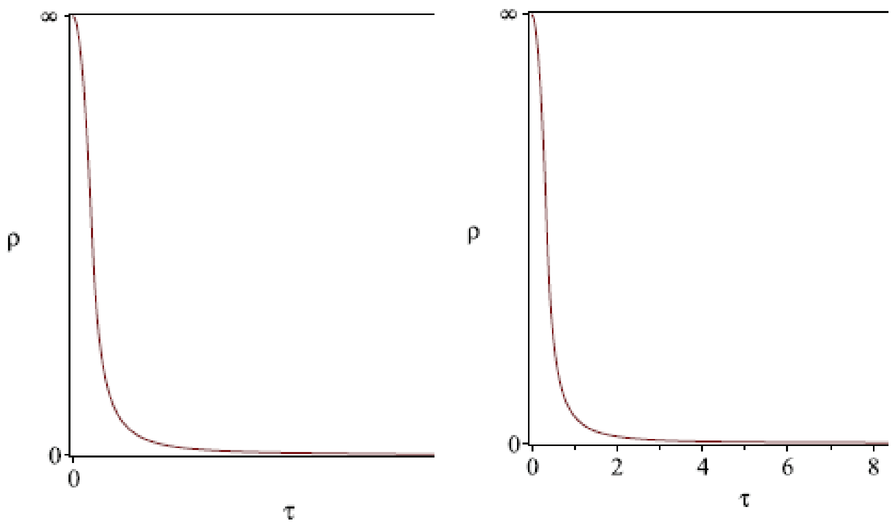
For massless scalar field the potential $V(\Phi)$ vanishes. Therefore, $p = \rho$ corresponds to stiff fluid (Madsen, 1988). Sound velocity in stiff fluid can be formulated as (Madsen, 1988):

$$c_s^2 = \frac{dp}{d\rho} c^2. \tag{60}$$

Sound velocity in stiff fluid is identical to velocity of light ($c_s = c$) for massless scalar field as expected. Stiff matter density yields

$$\rho = \frac{1}{2(\tau + c_6)^2 c_5^2}. \tag{61}$$

If we choose $c_5 \neq 0$ and $c_6 = 0$, stiff matter density has a singularity at $\tau = 0$ and converges to zero at $\tau \to \infty$ as depicted in Fig. 3.




(a) (b)

Fig. 3. Graphs of stiff matter density ($\rho$) with respect to time ($\tau$) for $c_5 = -1$, $c_6 = 0$, (a) $0 \leq \tau < \infty$ , (b) $0 \ \leq \ \tau \ \leq \ 10$.

## 4. CONCLUSION

We investigate unimodular $f(R,T)$ gravity by using Bianchi-I metric and massless scalar field. Unimodular form of Bianchi-I metric is ensured by using a time coordinate transformation. The Klein-Gordon equation and the field equations are solved together by using $f(R,T) = R + 2f_2(T)$ model. It is obtained $f(R,T)$ function, metric potentials and scalar potential functions. Obtained $f_2(T)$ function is consisted with $f_2(T) = \mu T$ form of similar works in the literature (Sharif and Zubair, 2012; Harko, Lobo, Nojiri and Odintsov, 2011). In this study, $\mu$ is found as a negative constant which corresponds to an effective cosmological constant. This situation is compatible with matter dominated era of the Universe.

Energy conditions (SEC and WEC) are satisfied for real scalar potential and metric potential functions. Graphics of energy conditions are plotted by choosing integration constants. This situation implies positive energy density and attractive gravity in a matter dominated universe.

Mean Hubble parameter and other kinematical quantities are obtained by according to both $\tau$ and $t$ time coordinates. Deceleration parameter is obtained as variable with $\tau$ coordinate and as positive constant with $t$ coordinate. Anisotropy parameter, shear and expansion scalar ratio show that unimodular Bianchi-I spacetime does not become isotropic at $\tau \to \infty$. Mean Hubble parameter corresponds to decelerating and matter dominated universe model. As it can be seen in Fig. 3 stiff matter density has initial singularity and decreases with time.

All these results are consisted with suitability between massless scalar field and stiff matter.

## 5. ACKNOWLEDGEMENTS

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# Chapter 5

## CONFORMAL COSMOLOGIES AS WORMHOLE GENERATORS IN F(R,T) THEORY

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## 1. INTRODUCTION

The search for exact solutions of Einstein field equation considering isotropic and anisotropic configurations has gain more interest in cosmology. There are great numbers of interior solutions including isotropic and anisotropic forms of matter of the gravitational field equations. Many of these works are concentrated isotropic case because astrophysical evidences promote isotropy in general. On the other hand, anisotropic objects in Einstein's gravity have been attracting attention for some time. Ruderman (1972) investigated that nuclear matter could be anisotropic for high density in the ranges of $\rho > 10^{15} g/cm^3$. Bowers and Liang (1974) examined relativistic fluid spheres for anisotropic equation of state considering modification of hydrostatic equilibrium through local anisotropy. They obtained that anisotropy could be affected on some parameters like maximum equilibrium mass and surface red-shift. On the other hand, there are well-known solutions which could be associated with anisotropic matter distribution such as Reissner-Nordström solution (Cho and Kim, 2019). In the solution, electromagnetic form of matter distribution exterior part of black hole corresponds to the anisotropy that provides pressure in two different directions. On the other hand, one of the most used methods is recognition of anisotropy for constitution of a compact star. It allows to some researchers to getting physically more suitable models, as well (Isayev, 2017; Pant et al., 2014; Mak and Harko, 2002; Mak and Harko; 2003).

Theoretical fundamental issues are reconsidered in detail through modified theories. *f(R,T)* is an inventive theory as one of the most important generalization of Einstein-Hilbert action. The theory is represented via a function comprising curvature scalar and trace of energy-momentum tensor (Harko et al., 2011). Einstein-Hilbert action of *f(R,T)* gravitation theory is described by Harko et al. (2011).

A solution of Einstein field equation refers to a wormhole which provides a connection for different part of space time regions. First example of the structure is procured by Einstein and Rosen (1935) as a new space-time metric when they disposed of some singularities under coordinate transformation for Schwarzschild solution. They get a solution which connect two spread sheets by a bridge (Einstein and Rosen, 1935). This new space-time metric was named as 'wormhole' for the first time by Misner and Wheeler (1957).

Structure of a wormhole is depicted with a tunnel having with a mouth and throat which is the minimal part of the tunnel (Morris and Thorne, 1988a; Morris and Thorne, 1988b). This throat can have constant or variable radius. Wormholes having throat with constant radius are called as static wormholes, while throat with variable radius are signed as dynamic wormholes (Morris and Thorne, 1988a; Morris and Thorne, 1988b). One of the most important issues in wormhole geometry is traversability condition. Morris and Thorne studied traversability (Morris and Thorne, 1988a) conditions of wormholes. They showed that the only possible way is to supplement wormhole with exotic matters in General Relativity. Such matter violates null energy condition. There are numerous constructed wormhole models with different exotic and/or normal matter forms such as quintom (Kuhfittig et al., 2010), scalar field models (Ford, 1987; Ratra and Peebles; 1988; Armendariz-Picon et al., 2000; Chiba et al., 2000) and electromagnetic field (Abdujabbarov and Ahmedov, 2009; Balakin et al., 2010) etc. in literature. Some important and interesting results are obtained by authors (Kuhfittig et al., 2010; Ford, 1987; Ratra and Peebles; 1988; Armendariz-Picon et al., 2000; Chiba et al., 2000). Also, conformal wormholes in General Relativity are studied primarily by Kar (1994). He shown that conformal wormholes could be consisted for a finite or half-infinite ($b < t \leq \infty$) interval of time. It satisfies weak energy condition. Kar and Sahdev (1996) studied energy conditions for conformal wormholes by considering different selections of conformal function.

Wormhole structures for different perspective are deeply examined in *f(R,T)* theory, as well. Azizi (2013) invented a shape function and shown that constructed wormhole solutions fulfilled the null energy condition in *f(R,T )* theory. Zubair et al. (2016) explored static wormhole structure and analyzed physical existence for different type of fluid distribution in *f(R,T)* theory. Moraes et al. (2017) studied on static wormholes. They shown that matter Lagrangian is governed by the total pressure of it. Many authors studied worm- hole geometries within the scope of *f(R,T)* theory (Sahoo et al., 2018; Moraes et al. 2019; Elizalde et al., 2018; Godani and Samanta, 2019).

Our aim is to investigation of conformal anisotropic spheres in *f(R,T)* theory. We would like to analyze different selection of conformal factor how effect matter form and geometrical structure of space-time in *f(R,T)* theory. This study organized as: In Section 2, *f(R,T)* theory is reintroduced. We set field equations of conformal anisotropic spheres in *f(R,T)* theory. We examined field equations for two different selection of conformal factor in *f(R,T)= R+2h(T)* models. Energy conditions for both effective and normal matters are studied for shape functions, as well. After, flaring out condition and embedding diagram of non-static conformal factor are investigated for constructed models. Last section, we discussed all properties of obtained wormhole models with a physical and geometric point of view.

## 2. $f(R,T)$ THEORY AND CONFORMAL COSMOLOGIES

Harko et al. (2011) offered Einstein-Hilbert action of *f(R,T)* theory as follows

$$S = \frac{1}{16\pi}\int f(R,T)\sqrt{-g}\, d^4x + \int \mathfrak{L}_{\mathrm{m}}\sqrt{-g}\, d^4x \qquad (1)$$

where $\mathfrak{L}_{\mathrm{m}}$ is Lagrangian density of cosmic matter form (Harko et al., 2011). *f(R,T)* is an arbitral function, allied with curvature scalar (*R*) and trace of energy momentum tensor (*T*). The energy-momentum tensor for cosmic matter is defined with $T_{ik} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathfrak{L}_{\mathrm{m}})}{\delta g^{ik}}$ (Landau and Lifshitz, 2002). The Lagrangian density of cosmic matter form depends on the metric tensor $g_{ik}$ (Sun and Huang, 2016) and dynamical components of the matter. Under this consideration, energy-momentum tensor is attained in the following form

$$T_{ik} = \mathfrak{L}_{\mathrm{m}} g_{ik} - 2\frac{\delta\mathfrak{L}_{\mathrm{m}}}{\delta g^{ik}} \qquad (2)$$

By considering variation of Eq. (1), one gets the field equations of *f(R,T)* theory in the following form:

$$f_R(R,T)R_{ik} - \frac{1}{2}f(R,T)g_{ik} + \left(g_{ik\backslash}Box - \nabla_i\nabla_k\right)f_R(R,T) = \kappa T_{ik} - f_T(R,T)T_{ik} \qquad (3)$$

$$-f_T(R,T)\Theta_{ik}$$

where

$$\Theta_{ik} = -\mathfrak{L}_{\mathrm{m}} g_{ik} + 2g_{ik\mathfrak{L}_{\mathrm{m}}} - 2T_{ik} - 2g^{ml}\frac{\partial^2\mathfrak{L}_{\mathrm{m}}}{\partial g^{ik}\,\partial g^{ml}} \qquad (4)$$

Here $f_R(R,T) = \frac{\partial f(R,T)}{\partial R}$ and $f_T(R,T) = \frac{\partial f(R,T)}{\partial T}$ (Harko et al., 2011). □ identifies d'Alambertian operator and $\nabla_{\mathrm{i}}$ performs covariant derivative (Harko et al., 2011). It is possible to get a following connection through contraction of Eq. (3)

$$f_R(R,T)R + 3\Box f_R(R,T) - 2f(R,T) = \kappa T - f_T(R,T)(T+\Theta) \quad (5)$$

where

$$\Theta_{ik} = g^{ik}\frac{\partial T_{ik}}{\partial g^{ik}} \quad (6)$$

In order to avoid the vanishing of the extra force, matter Lagrangian could be chosen as $\mathfrak{L}_{\mathfrak{m}} = -\rho$ (Elizalde and Khurshudyan, 2019). Different selections of cosmic matter source have impact on characteristic of *f(R,T)* theory. In theory, Harko et al. (2011) offered some classification of *f(R,T)* functions. One of the classification of *f(R,T)* functions is

$$f(R,T) = R + 2h(T) \quad (7)$$

where $h(T) = \lambda T$ (Harko et al., 2011). The field equations of the *f(R,T)* theory could be rewritten by using Eq. (7) (Harko et al., 2011; Elizalde and Khurshudyan, 2019):

$$G_{ik} = (8\pi + 2\lambda)T_{ik} + \lambda(2\rho + T)g_{ik} \quad (8)$$

where $G_{ik}$ is called as Einstein tensor.

We consider spherically symmetric conformal metric as follows:

$$ds^2 = \Omega^2(t)\left[dt^2 - \frac{dr^2}{B(r)} - r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)\right] \quad (9)$$

where $\Omega^2(t)$ is the conformal factor and it is considered as finite and strictly positive function (Wald, 1984). Matter content of the universe is assumed as anisotropic fluid. Energy-momentum tensor for anisotropic fluid is described as

$$T_{ik} = (\rho + p_t)u_i u_k - p_t g_{ik} + (p_r - p_t)\chi_i \chi_k \quad (10)$$

where ρ, $p_t$ and $p_r$ are energy density, perpendicular (to the inhomogeneous direction) pressure, and parallel pressure respectively as measured in the fluid elements rest frame. $u_i$ is four-velocity in co-moving coordinates and $\chi_i$ is a space-like vector orthogonal to $u^i$.

Energy conditions in wormholes physics are an important fundamental issue in General Relativity. In General Relativity, non-vacuum solutions of wormholes don't satisfy all energy conditions if the wormholes are filled with exotic matters (Bhatti et al., 2018). Therefore, investigation of wormhole structure in other theories cause to studied on energy conditions, as well. Null Energy Condition (NEC), Weak Energy Condition (WEC), Dominant Energy Condition (DEC) and Strong Energy Condition (SEC) for anisotropic fluid in *f(R,T)* theory are given by (Bhatti et al., 2018):

$$NEC\ :\ \rho^{eff}+p_i^{eff}\geq 0 \tag{11}$$

$$WEC\ :\ \rho^{eff}\geq 0;\ \ \rho^{eff}+p_i^{eff}\geq 0 \tag{12}$$

$$DEC\ :\ \rho^{eff}\geq 0;\ \ \rho^{eff}\geq \left|p_i^{eff}\right| \tag{13}$$

$$SEC\ :\ \rho^{eff}+p_i^{eff}\geq 0;\ \ \rho^{eff}+p_r^{eff}+2p_t^{eff}\geq 0 \tag{14}$$

By using Eqs. (8), (9) and (10), we get field equations for conformal wormholes with anisotropic fluid in *f(R,T)* theory as follows:

$$\frac{2\Omega''}{\Omega^3}-\frac{\Omega'^2}{\Omega^4}-\frac{1-B}{\Omega^2 r^2}=\lambda(3\rho-2p_t)-p_r(8\pi+3\lambda)=-p_r^{eff} \tag{15}$$

$$\frac{2\Omega''}{\Omega^3}-\frac{\Omega'^2}{\Omega^4}-\frac{\dot{B}}{2\Omega^2 r}=\lambda(3\rho-p_r)-p_t(8\pi+4\lambda)=-p_t^{eff} \tag{16}$$

$$\frac{3\Omega'^2}{\Omega^4}-\frac{\dot{B}}{\Omega^2 r}-\frac{1-B}{\Omega^2 r^2}=-\lambda(2p_t+p_r)+\rho(8\pi+5\lambda)=\rho^{eff} \tag{17}$$

where prime denotes derivative with respect to cosmic time while dot represents derivative with respect to radial coordinate. In Eqs. (15)-(17), unknown quantities for matter distribution are attained as

$$p_r(r,t)=\frac{-\Omega''\Omega r^2(5\lambda+8\pi)+\Omega'^2r^2(7\lambda+4\pi)-2\Omega^2\left(\dot{B}\lambda r-(\lambda+2\pi)(B-1)\right)}{8\Omega^4r^2(\lambda+\pi)(\lambda+4\pi)} \tag{18}$$

$$p_t(r,t)\frac{-\Omega''\Omega r^2(5\lambda+8\pi)+\Omega'^2r^2(7\lambda+4\pi)-2\Omega^2\left(\dot{B}\pi r-\lambda(B-1)\right)}{-\Omega''\Omega r^2(5\lambda+8\pi)+\Omega'^2r^2(7\lambda+4\pi)-2\Omega^2(\dot{B}\pi r-\lambda(B-1)\}8\Omega^4r^2(\lambda+\pi)(\lambda+4\pi)} \tag{19}$$

$$\rho(r,t)=\frac{-3\Omega''\Omega\lambda r^2+3\Omega'^2r^2(3\lambda+4\pi)-2\Omega^2(\lambda+2\pi)(\dot{B}r+B-1)}{8\Omega^4r^2(\lambda+\pi)(\lambda+4\pi)} \tag{20}$$

We obtain that Eqs. (15)-(17) have two different common solutions, static and non-static, as presented below.

### 2.1 Constant Conformal Factor

In this part, we consider conformal factor as constant. By considering Eqs. (15)-(17) with constant conformal factor, we get all unknown geometrical and matter components:

$$B(r)=1-\sqrt{\frac{c_1}{r}} \tag{21}$$

$$\Omega(t) = e^{c_2} \tag{22}$$

$$\rho(r) = \frac{1}{8}\frac{e^{-2c_2\sqrt{c_1}}(\lambda+2\pi)}{r^{\frac{5}{2}}(\lambda+\pi)(\lambda+4\pi)} \tag{23}$$

$$p_r(r) = -\frac{1}{8}\frac{e^{-2c_2\sqrt{c_1}}(3\lambda+4\pi)}{r^{\frac{5}{2}}(\lambda+\pi)(\lambda+4\pi)} \tag{24}$$

$$p_t(r) = \frac{1}{8}\frac{e^{-2c_2\sqrt{c_1}}(2\lambda+\pi)}{r^{\frac{5}{2}}(4\pi^2+5\pi\lambda+\lambda^2)} \tag{25}$$

From, Eqs. (23)-(25), it is seen that anisotropic fluid loses its time dependence. The line element for constructed model could be rewritten static form:

$$ds^2 = e^{2c_2}\left[-dt^2 + \frac{dr^2}{1-\sqrt{\frac{c_1}{r}}} + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)\right] \tag{26}$$

Obtained line element could be associated with wormhole shape function studied by Lobo and Oliveira (2009) in *f(R)* theory. In the case of $c_1 = r_0$, effective matter density, parallel and perpendicular pressure are attained from Eqs. (15)-(17) with (21)-(25):

$$\rho^{eff}(r) = \frac{1}{2}\frac{e^{-2c_2\sqrt{r_0}}}{r^{\frac{5}{2}}} \tag{27}$$

$$p_r^{eff}(r) = -\frac{e^{-2c_2\sqrt{r_0}}}{r^{\frac{5}{2}}} \tag{28}$$

$$p_t^{eff}(r) = \frac{1}{4}\frac{e^{-2c_2\sqrt{r_0}}}{r^{\frac{5}{2}}} \tag{29}$$

Effective matter density is physically meaningful in all region for the model. Also, $\rho^{eff} + p_r^{eff} \geq 0$ condition related with NEC is violated in all region. It means that energy conditions such as NEC, WEC and SEC are violated due to negative value of parallel pressure, $p_r$. Also, energy condition term $\rho^{eff} \geq |p_r^{eff}|$ related with DEC is violated in all region, as well. Considering energy conditions for anisotropic matter distribution, all terms are gained as follows:

$$\rho + p_r = -\frac{1}{4}\frac{e^{-2c_2\sqrt{r_0}}}{r^{\frac{5}{2}}(\lambda+4\pi)} \tag{30}$$

$$\rho + p_t = \frac{3}{8}\frac{e^{-2c_2\sqrt{r_0}}}{r^{\frac{5}{2}}(\lambda+4\pi)} \tag{31}$$

$$\rho + p_r + 2p_t = \frac{1}{4}\frac{\lambda e^{-2c_2\sqrt{r_0}}}{r^{\frac{5}{2}}(4\pi^2+5\pi\lambda+\lambda^2)} \quad (32)$$

$$\rho - |p_r| = \frac{1}{8}\frac{e^{-2c_2\sqrt{r_0}}(\lambda+2\pi)}{r^{\frac{5}{2}}(\lambda+\pi)(\lambda+4\pi)} - \left|-\frac{1}{8}\frac{e^{-2c_2\sqrt{r_0}}(3\lambda+4\pi)}{r^{\frac{5}{2}}(\lambda+\pi)(\lambda+4\pi)}\right| \quad (33)$$

$$\rho - |p_t| = \frac{1}{8}\frac{e^{-2c_2\sqrt{r_0}}(\lambda+2\pi)}{r^{\frac{5}{2}}(\lambda+\pi)(\lambda+4\pi)} - \left|\frac{1}{8}\frac{e^{-2c_2\sqrt{r_0}}(2\lambda+\pi)}{r^{\frac{5}{2}}(4\pi^2+5\pi\lambda+\lambda^2)}\right| \quad (34)$$

The change of energy condition terms are given by Figs. 1-6. In Fig. 1, evolution of energy density is violated when $\lambda = -5\pi$. In order to get a physically suitable model in terms of energy density, it must be selected as $\lambda > \pi$ or $-2\pi > \lambda > -4\pi$. Evolution of energy condition term related with SEC $(\rho + p_r + 2p_t)$ is represented In Fig. 2. SEC for constructed model is satisfied in the cases of $\lambda < -4\pi$ or $\lambda > \pi$. In Fig. 3 and 4, evolution of energy condition terms $\rho + p_r$ and $\rho + p_t$ are plotted with respect to different selections of $\lambda$, as well. $\rho + p_r \geq 0$ condition related with NEC, WEC and SEC could be satisfied in the case of $\lambda < -4\pi$, while $\rho + p_t \geq 0$ condition could be satisfied in the case of $\lambda > 4\pi$. It is obviously seen that $\rho + p_i \geq 0$ condition can not be satisfied both parallel and perpendicular pressure for same selection of $\lambda$ and it causes to violation of NEC, WEC and SEC for constructed model. Also, evolution of energy condition terms $\rho - |p_r|$ and $\rho - |p_t|$ are represented in Figs. 5 and 6. DEC for constructed model is violated because of $\rho - |p_r| \geq 0$ term.

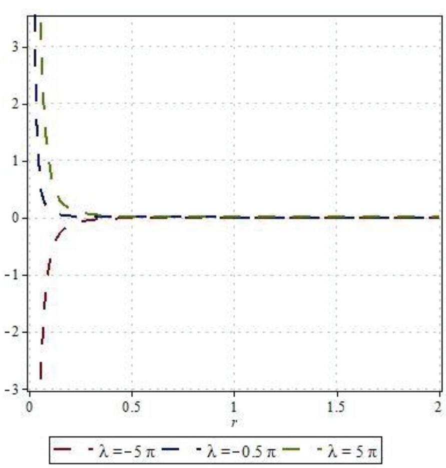


Figure 1: Evolution of density for $c_2$ =and $r_0 = 0.5$

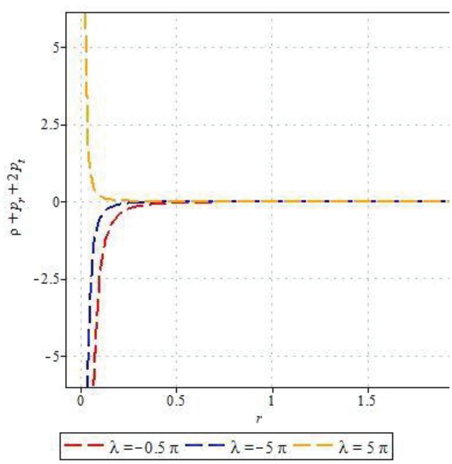


Figure 2: Evolution of energy condition term $\rho+p_r+2p_t$ for $c_2 = 1.1$ and $r_0 = 0.5$

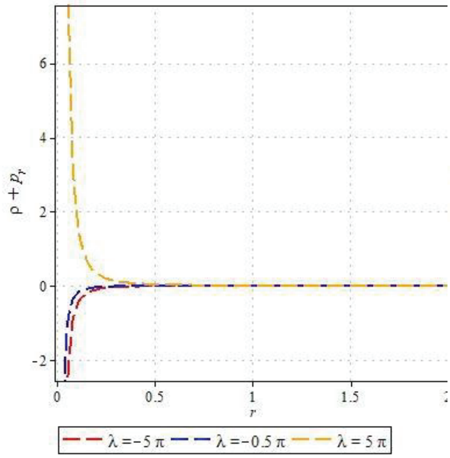


Figure 3: Evolution of energy condition term ρ + pr for c2 = 1.1 and r0 = 0.5

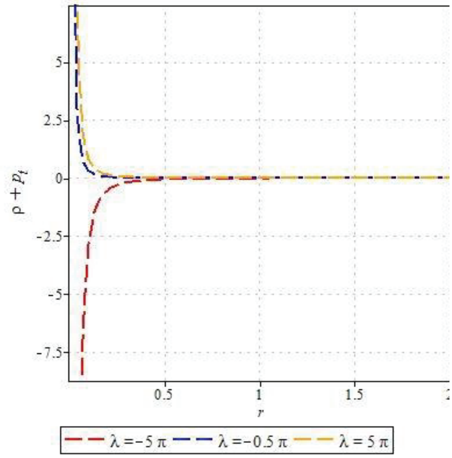


Figure 4: Evolution of energy condition term ρ + pt for c2 = 1.1 and r0 = 0.5

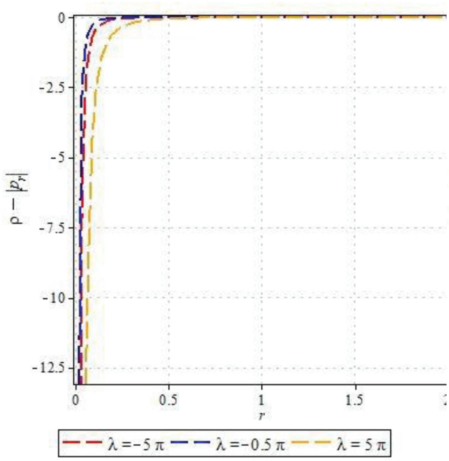


Figure 5: Evolution of energy condition term ρ − |pr| for c2 = 1.1 and r0 = 0.5

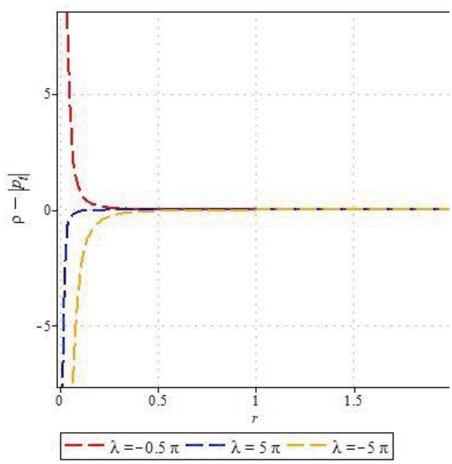


Figure 6: Evolution of energy condition term ρ − |pt| for c2 = 1.1 and r0 = 0.5

### 2.1 Non-Static Conformal Cosmologies

In this part, we consider conformal factor as non-static form. By considering Eqs. (15)-(17) with non-constant conformal factor, we get all unknown geometrical and matter components as follows:

$$B(r) = 1 - c_3^2 r^2 \tag{35}$$

$$\Omega(t) = e^{c_4} cos^{-\frac{1}{c_6}}(c_3(c_5 + t)c_6) \quad (36)$$

$$\rho(t) = \frac{3}{8}\frac{c_3^2 e^{-2c_4}(4\pi+\lambda(2-c_6))cos^{-\frac{2(c_6-1)}{c_6}}(c_3(c_5+t)c_6)}{(\pi+\lambda)(4\pi+\lambda)} \quad (37)$$

$$p_r(t) = p_t(t) = -\frac{1}{8}\frac{c_3^2 e^{-2c_4}(4\pi(2c_6+1)+\lambda(5c_6-2))cos^{-\frac{2(c_6-1)}{c_6}}(c_3(c_5+t)c_6)}{(\pi+\lambda)(4\pi+\lambda)} \quad (38)$$

From, Eqs. (37)-(38), it is obtained that anisotropic fluid loses its radial coordinate dependence. In this case, anisotropic fluid form turns to isotropic fluid form. The line element is rewritten as:

$$ds^2 = e^{2c_4} cos^{-\frac{1}{c_6}}(c_3(c_5 + t)c_6)\left[-dt^2 + \frac{dr^2}{1-c_3^2 r^2} + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)\right] \quad (39)$$

Obtained line element could be associated with wormhole shape function studied by Zubair (2016) in *f(R,T)* theory. In the case $c_3 = r_0$, constructed model leads to wormhole structure. Under this consideration, effective matter density, radial and tangential pressure are attained from Eqs. (15)-(17) with (37) and (38) as:

$$\rho^{eff}(t) = 3e^{-2c_4} r_0^2 cos^{-\frac{2(c_6-1)}{c_6}}(r_0(c_5 + t)c_6) \quad (40)$$

$$p^{eff}(t) = -(2c_6 + 1)e^{-2c_4} r_0^2 cos^{-\frac{2(c_6-1)}{c_6}}(r_0(c_5 + t)c_6) \quad (41)$$

Effective matter density is physically meaningful in the cases $-\frac{3}{2}\pi \geq r_0 c_6 \geq -\frac{5}{2}\pi$, $0 \geq r_0 c_6 \geq -\frac{1}{2}\pi$, $\frac{1}{2}\pi \geq r_0 c_6 \geq 0$ or $\frac{5}{2}\pi \geq r_0 c_6 \geq \frac{3}{2}\pi$ when $t > c_5$. All energy conditions given by (11)-(14) considering for isotropic fluid could be satisfied in the case of $\frac{1}{2}\pi > c_5 + t > -\frac{1}{2}$ or $\frac{3}{2}\pi > c_5 + t > \frac{1}{2}\pi$ when $c_6 = -1$. Considering energy conditions for normal matter distribution, all terms are gained as follows:

$$\rho + p = -\frac{r_0^2 e^{-2c_4}(c_6-1)cos^{-\frac{2(c_6-1)}{c_6}}(r_0(c_5+t)c_6)}{4\pi+\lambda} \quad (42)$$

$$\rho + 3p = -\frac{3}{4}\frac{r_0^2 e^{-2c_4}(\lambda(3c_6-2)+4\pi c_6)cos^{-\frac{2(c_6-1)}{c_6}}(r_0(c_5+t)c_6)}{(\pi+\lambda)(4\pi+\lambda)} \quad (43)$$

$$\rho - |p| = \frac{3}{8}\frac{r_0^2 e^{-2c_4}\left(4\pi+\lambda(2-c_6)\right)\cos^{-\frac{2(c_6-1)}{c_6}}\left(r_0(c_5+t)c_6\right)}{(\pi+\lambda)(4\pi+\lambda)} - \left|-\frac{1}{8}\frac{r_0^2 e^{-2c_4}\left(4\pi(2c_6+1)+\lambda(5c_6-2)\right)\cos^{-\frac{2(c_6-1)}{c_6}}\left(r_0(c_5+t)c_6\right)}{(\pi+\lambda)(4\pi+\lambda)}\right| \quad (44)$$

The change of energy condition terms with respect to time and $\lambda$ constant are given by in Figs. 7-10. In Fig. 7, energy density could be get positive values under $\lambda > 0$ and $c_6 \leq \frac{2(2\pi+\lambda)}{\lambda}$. Evolution of energy condition term $\rho + p$ related with NEC is represented In Fig. 8. It is obviously seen that some selections of arbitrary constant have effect on stability of condition. In the cases of $1 \geq c_6 \geq ,0 \geq c_6 \geq -\frac{1}{2}\pi$ or $-\frac{3}{2}\pi \geq c_6 \geq -\frac{5}{2}\pi$ for $\lambda > -4\pi$, $\rho + p \geq 0$ condition could be valid. It means that wormhole structure satisfies NEC. Also, change of energy condition term $\rho + 3p$ correlated with SEC is plotted in Fig. 9. Under selection of consideration with $-\frac{3}{2}\pi \geq c_6 \geq -\frac{5}{2}\pi$, $0 \geq c_6 \geq -\frac{1}{2}\pi$ or $\frac{2\lambda}{4\pi+\lambda} > c_6 > 0$ with $\lambda > 0$, SEC condition is valid through $\rho + 3p \geq 0$ . In Fig. 10, it is clearly seen that DEC related with $\rho - |p| \geq 0$ could be satisfied under some consideration such as $\lambda > 0$ and $c_6 = \frac{2\pi}{4\pi+3\lambda}$, as well. Under some special selections of arbitrary constant, both SEC and DEC can not be provided in the same definition range when WEC and NEC is satisfied in all time when $c_6 = 1$. Also, in selections of $c_6 = -1$ and $\lambda = -\frac{4}{5}$, isotropic fluid form behaves as string gas form which is corresponded to $\omega = -\frac{1}{3}$ in Equation of State (Kamenshchik and Khalatnikov, 2012).

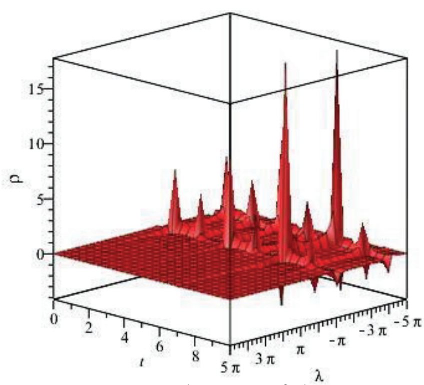


Figure 7: Evolution of density for c4 = 1,c5 = 0, c6 = 2 and r0 = 1

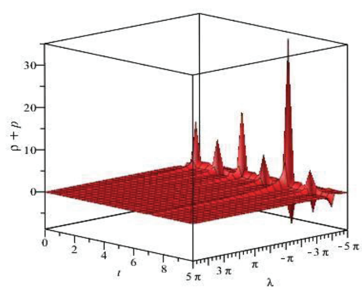


Figure 8: Evolution of energy condition term ρ + p for c4 = 1,c5 = 0, c6 = 2 and r0 = 1

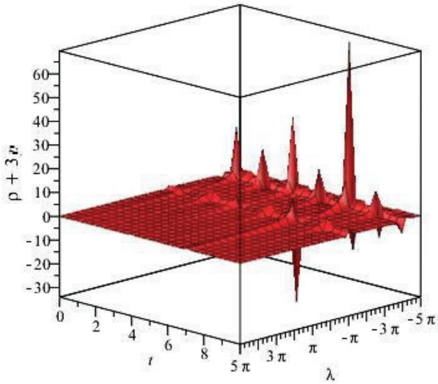


Figure 9: Evolution of density for ρ +3p for c4 = 1,c5 = 0, c6 = 2 and r0 = 1

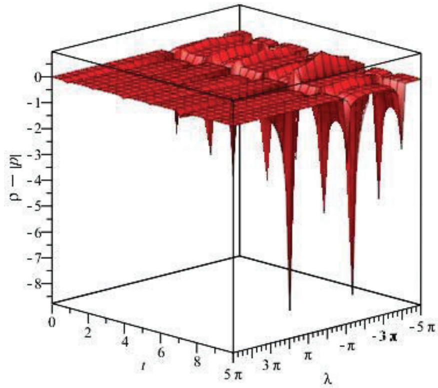


Figure 10: Evolution of energy condition term ρ −|p| for c4 = 1,c5 = 0, c6 = 2 and r0 = 1

It is possible to verify embedding diagram and flaring out condition of the obtained wormholes. We consider an equatorial slice $\theta = \frac{\pi}{2}$ and cosmic time as a constant in order to fixed moment (Arellano and Lobo, 2006; Arellano et al., 2009). Under consideration metric form of wormhole structure turns as

$$ds^2 = \Omega(t)^2\left(\frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\phi^2\right) \tag{45}$$

It is possible to embedding to metric form given by Eq. (45) in 3-dimensional Euclidean space

$$ds^2 = d\bar{z}^2 + d\bar{r}^2 + \bar{r}^2 d\phi^2 \tag{46}$$

by using following relationships (Arellano and Lobo, 2006; Arellano et al., 2009)

$$\bar{r} = \Omega(t) r|_{t=const}, \quad d\bar{r}^2 = \Omega(t)^2 dr^2|_{t=const} \tag{47}$$

Metric form of wormhole could be rewritten by considering $\bar{z}$, $\bar{r}$ and $\phi$ coordinates as follows

$$ds^2 = \frac{d\bar{r}^2}{1-\frac{\bar{b}(\bar{r})}{\bar{r}}} + \bar{r}^2 d\phi^2 \tag{48}$$

Where $\bar{b}(\bar{r})$ has a minimum at $\bar{b}(\bar{r}_0) = \bar{r}_0$ (Arellano and Lobo, 2006; Arellano et al., 2009). Also, Eq. (48) could be turned to Eq. (45) by using relationships in Eq. (47) with the definition of $\bar{b}(\bar{r}) = \Omega(t) b(r)$ (Arellano and

Lobo, 2006; Arellano et al., 2009). One of the fundamental condition in wormhole physics is flaring out condition. The condition makes a sign that radial coordinate must have a regular minimum at the throat. By considering above motivation, flaring out condition could be written at or near the throat in the following form (Arellano and Lobo, 2006; Arellano et al., 2009):

$$\frac{d\bar{r}^2(\bar{z})}{d\bar{z}^2} = \frac{1}{\Omega(t)}\frac{1-\dot{b}r}{2b^2} = \frac{1}{\Omega(t)}\frac{d^2r(z)}{dz^2} > 0. \tag{49}$$

The condition is satisfied in the range of

$$-\frac{1}{e^{c_4}r_0^2r^3\cos^{-\frac{1}{c_6}}(r_0(c_5+t)c_6)} > 0. \tag{50}$$

The condition could be satisfied depending on sign of trigonometric function. In the case of $c_5 > t$, inside terms of trigonometric function must be selected as $\frac{3}{2}\pi > r_0(c_5+t)c_6 > \frac{1}{2}$ or $-\frac{1}{2}\pi > r_0(c_5+t)c_6 \succ \frac{3}{2}$. In order to get rid of complex value of trigonometric function, $c_6$ must be selected as $c_6 = \pm 1$. Under all consideration, the flaring out condition is valid for negative values of $\Omega$. Embedding diagram related with $\bar{z}(\bar{r})$ could be investigated with following equation:

$$\frac{d\bar{z}(\bar{r})}{d\bar{r}} = \frac{dz(r)}{dr} = \pm\left(\frac{r}{b}-1\right)^{-\frac{1}{2}}. \tag{51}$$

From Eq. (51), we get $z(r)$ function:

$$z(r) = \frac{\sqrt{1-r_0^2r^2}}{r_0}. \tag{52}$$

Three-dimensional graphical illustration of the spherically symmetric space can be reviewed as proper cylindrical coordinates with $x = r\, cos\phi$, $y = r\, sin\phi$ and $z = z(r)$ in Figs. 11 and 12.

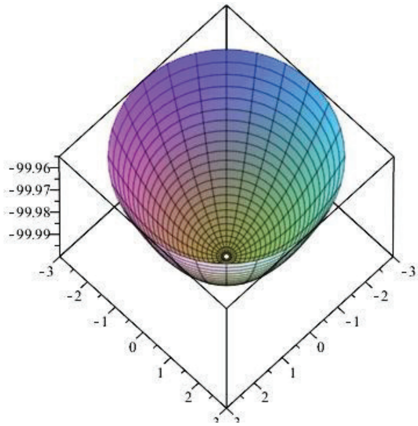


Figure 11: Embedding diagram top view of constructed wormhole model in the cases of r0 = 0.01 and 0 < r < 3.

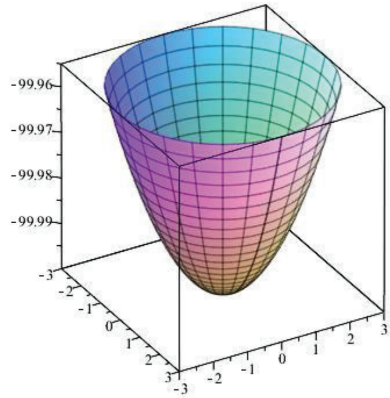


Figure 12: Embedding diagram front view of constructed wormhole model in the cases of r0 = 0.01 and 0 < r < 3.

Wormhole structure given by Eqs. (39) has similar cosmology with FRW-like wormholes for $k = 0$ (Li-Xin, 2001). Ω function corresponds to the scale factor in FRW-like wormholes. Under this consideration, it is possible to get expansion velocity and expansion acceleration by the way of Ω function as follows:

$$\Omega'(t) \propto e^{c_4} r_0 cos^{-\frac{c_6+1}{c_6}}(r_0(c_5+t)c_6) sin(r_0(c_5+t)c_6) \tag{53}$$

$$\Omega''(t) \propto -e^{c_4} r_0^2 cos^{-\frac{2c_6+1}{c_6}}(r_0(c_5+t)c_6)(cos^2(r_0(c_5+t)c_6) - c_6 - 1). \tag{54}$$

The change of expansion velocity and expansion acceleration for the wormhole model are illustrated with respect to cosmic time in Fig 13 and 14. It is seen that the wormhole model changes on different ranges of time. The results show an oscillating wormhole model. Both expansion velocity and expansion acceleration supports the idea about oscillation action.

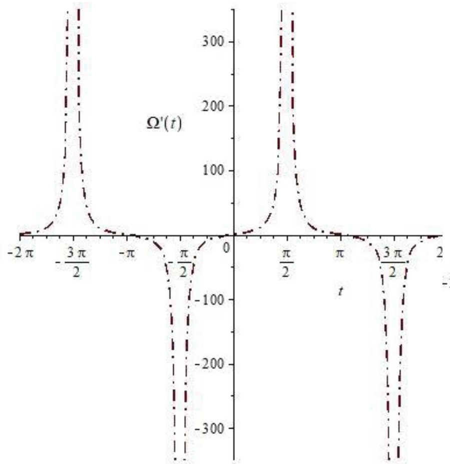


Figure 13: Evolution of expansion veloc- ity for constructed wormhole model in the cases of c4 = 2, c5 = 0, c6 = 1 and r0 = 1.

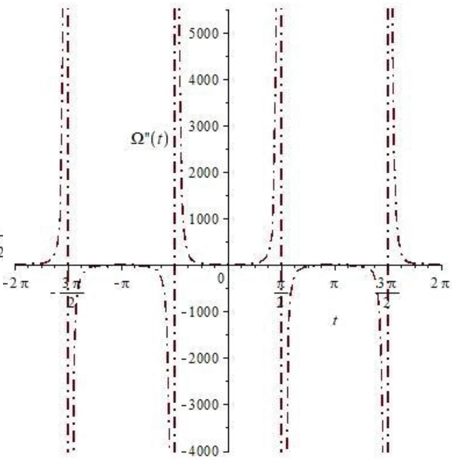


Figure 14: Evolution of expansion accel- eration for constructed wor- mhole model in the cases of c4 = 2, c5 = 0, c6 = 1 and r0 = 1.

## 3. CONCLUSION

In this study, we investigate conformal anisotropic spheres in *f(R,T)* theory. We obtain field equations of conformal anisotropic spheres for *f(R,T)=R+2h(T)* model. Constructed field equations are examined for two different conformal factor case.

In first case, field equations lead to constant conformal factor. Also, anisotropic fluid loses its time dependence. But, there is an interesting result in this case. So, obtained model leads to a shape function, $b(r) = \sqrt{r_0 r}$ was studied by Lobo and Oliviera (2009) in *f(R)* theory. Lobo and Oliviera attained a wormhole solution for $b(r) = \sqrt{r_0 r}$ function by considering specific equation of state, $p_t = \alpha$ in *f(R)* theory. They shown that obtained model is satisfied to WEC for particular selections of constants from the solutions. In our study, energy conditions for cosmic matter form are examined for wormhole structure, as well. WEC, NEC and SEC for wormhole structure are violated because $\rho + p_i \geq 0$ condition can not be satisfied both parallel and perpendicular pressure for same selection of λ. Also, DEC is violated, as well. The effective matter components are also attained for constructed model. Effective matter components lose their theory constant dependence, λ. It is obviously seen that $\rho^{eff}$ and $p_t^{eff}$ are positive while $p_r^{eff}$ is negative in all region. All energy conditions for effective matter components are violated for constructed model.

In second case, field equations lead to non-static conformal factor. Also noteworthy at this point is that, similar to the constant conformal factor case, obtained model exhibits wormhole feature. The model leads to a shape function, $b(r) = r_0^2 r^3$, was studied by Zubair et al. (2016) in *f(R,T)* theory. They attained a solution for different selection of shape function in *f(R,T)* theory. They shown that a specific choose of shape function, $b(r) = r_0^2 r^3$, render possible to satisfied all condition for some specific selection of solution constants without losing its anisotropic properties. In this study, energy conditions for cosmic matter form are examined for constructed wormhole structure, as well. All energy conditions for wormhole structure are satisfied by depending on some different selection of theory constant, λ, and arbitrary constant. Also, three-dimensional graphical representation of wormhole structure is represented, as well. Under some special selections of arbitrary constant, both SEC and DEC can not be provided in the same definition range. On the other hand, WEC and NEC are satisfied in all time for the model when $c_6 = 1$. Also, anisotropic fluid turns to isotropic fluid form with losing its radial coordinate dependence. Also, effective matter components are attained as trigonometric form. Effective matter components lose their theory constant dependence, λ, as well. The effective matter form turns to dust matter form when $c_6 = \frac{1}{2}$ while density and pressure of matter form are constants in the case of $c_6 = 1$. Flaring out condition is investigated for constructed model. In order to avoid complex value of Ω, arbitrary constant must be selected as $c_6 = \pm 1$. In the condition of $c_6 = 1$, matter form behaves as exotic matter with constant pressure and density while matter form does not lose its time dependence for $c_6 = -1$. Also, in selections of $c_6 = -1$ and $\lambda = -\frac{4}{5}$, isotropic fluid form behaves as string gas form which is corresponded to $\omega = -\frac{1}{3}$ in Equation of State (Kamenshchik and Khalatnikov, 2012). It is seen that static and non-static wormhole models obtained in this study have same oscillating features as expanding and non-expanding wormholes investigated within the scope of Lyra geometry (Doğru and Yılmaz, 2015).

The fact that both solutions obtained from field equations give rise to wormhole geometry raises question whether this may be a reason for generalization in *f(R,T)* theory for conformal symmetry. In order to find correct answer to the question, it is necessary to consider other conformal symmetry studies investigated within the scope of *f(R,T)* theory. In the literature, there is no study that indicates a wormhole by examining distribution of matter for direct conformal symmetric line element, which is similar to our study. However, Banerjee et al. (2020) proved that static and Lorentzian wormholes in *f(R,T)* theory have traversable features under Conformal Killing Symmetry for the $f(R,T)=R+2\lambda T$ model. A generalization of the work under conformal symmetry defined in the presence of homothetic vector was ob- tained by Zubair et al. (2019). They showed that anisotropic fluid allows static wormholes in non-comutative geometry in *f(R,T)* theory. Bhar et al. (2022) showed that phantom energy under conformal symmetry would allow traversable static Lorentzian wormholes. These investigations are kind of special cases of the first case with a constant conformal factor in this study. They support to the conclusion that conformal symmetry could be generators of wormhole geometry. In addition, Sharif and Waseem (2019), Bhar and Rej (2021), Das et al. (2016) presented studies associating conformal symmetry with compact stars and/or gravastars in *f(R,T)* theory. It is seen that the obtained solutions allow black hole and similar geometries. It is understood that the solutions in question can be suitable for the wormhole geometry by adjusting the singular point settings or under various coordinate transformations. Based on the results obtained both in the literature and within the scope of this study, it can be said that conformal symmetry may be the generators of a wormhole.

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